Kernel Theorems in Coorbit Theory

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joint work with:
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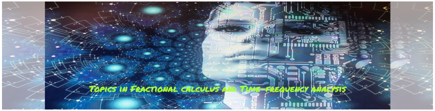
Topics in Fractional calculus and Time-frequency analysis

Dedicated to the memory of Professor Arpad Takači



Rationale of Kernel Theorems:

"Reasonable" operators can written as "generalized" integral operators.

- Schwartz kernel theorem:[1]
  $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ continuous $\Leftrightarrow \exists!$ kernel $K \in \mathcal{S}'(\mathbb{R}^{2d})$, s.t.

$$\langle A\upsilon, \varphi \rangle = \langle K, \varphi \otimes \upsilon \rangle, \ \upsilon, \varphi \in \mathcal{S}(\mathbb{R}^d) \, .$$

- More distribution spaces: e.g. Gelfand-Shilov spaces.[2]

[1] L. Hörmander. The Analysis Of Linear Partial Differential Operators. Springer New York, 1989.

[2] I. M. Gel'fand and G. E. Shilov. Generalized functions. Vol. 2. Spaces of fundamental and generalized functions, Translated from the 1958 Russian original [MR0106409] by Morris D. Friedman, Amiel Feinstein and Christian P. Peltzer, Reprint of the 1968 English translation [MR0230128]. AMS Chelsea Publishing, Providence, RI, 2016, pp. x+261. ISBN: 978-1-4704-2659-0.

- Feichtinger's kernel theorem for modulation spaces[3].[4]
  - Advantage: Banach spaces.
  - Associate a proper integral operator $\Rightarrow$ Schur's test.
- **Generalization to general coorbit spaces.**[5]

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[3] H. G. Feichtinger. "Un espace de Banach de distributions tempérées sur les groupes localement compacts abéliens". In: C. R. Acad. Sci. Paris Sér. A-B 290.17 (1980), A791–A794. ISSN: 0151-0509.

[4] E. Cordero and F. Nicola. "Kernel theorems for modulation spaces". In: Journal of Fourier Analysis and Applications (2017). DOI: https://doi.org/10.1007/s00041-017-9573-3.

[5] Peter Balazs, Karlheinz Gröchenig, and Michael Speckbacher. "Kernel theorems in coorbit theory". In: Transactions of the American Mathematical Society Series B 6 (2019), pp. 346–364.

Group Representations

Setup:

- G locally compact group
- dg left Haar measure
- $\mathcal{H}$ separable Hilbert space
- $\pi : G \to \mathcal{U}(\mathcal{H})$ unitary representation

π square-integrable: π is irreducible and there exist $\psi \in \mathcal{H} \backslash \{0\}$ s.t.

$$\int_G |\langle \psi, \pi(g)\psi \rangle|^2 dg < \infty .$$

$\psi \neq 0$ is called **admissible**
There exists T densely defined s.t. $\forall f_1, f_2 \in \mathcal{H},\ \psi_1, \psi_2 \in \operatorname{dom}(T)$:

$$\int_G \langle f_1, \pi(g)\psi_1 \rangle \langle \pi(g)\psi_2, f_2 \rangle dg = \langle T\psi_2, T\psi_1 \rangle \langle f_1, f_2 \rangle .$$

Group Representations

- **Generalized wavelet transform:**

$$V_\psi f(g) := \langle f, \pi(g)\psi \rangle, \qquad f \in \mathcal{H},\ \psi \in \mathsf{dom}\,(T)$$

- Assume w.l.o.g. $\|T\psi\| = 1 \Rightarrow V_\Psi$ isometry
- $V_\psi^* : L^2(G) \to \mathcal{H}$ is given by

$$V_\psi^* F := \int_G F(g)\pi(g)\psi dg, \qquad F \in L^2(G)$$

- $I = V_\psi^* V_\psi$

$$f = \int_G \langle f, \pi(g)\psi \rangle \pi(g)\psi dg, \qquad f \in \mathcal{H}$$

Coorbit Theory

Rationale of Coorbit Theory:

Functions are "nice" $\Leftrightarrow$ their generalized wavelet transform is "nice".

To measure "nice" one uses weighted L^p-spaces on G:

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To measure "nice" one uses weighted L^p-spaces on G:
Let $g_1, g_2, g_3 \in G$. We assume:

- $w : G \to \mathbb{R}^+$ **submultiplicative**, i.e., $w(g_1 g_2) \leqslant w(g_1) w(g_2)$
- $m : G \to \mathbb{R}^+$ **w-moderate**, i.e., $m(g_1 g_2 g_3) \leqslant w(g_1) m(g_2) w(g_3)$
- Some further technical assumptions on w

π **integrable w.r.t. w:** there exists ψ admissible s.t.

$$\psi \in \mathcal{A}_w(G) := \left\{ \psi \in \mathcal{H}, \psi \neq 0 : \ V_\psi \psi \in L^1_w(G) \right\}.$$

Test function space: $\mathcal{H}^1_w := \left\{ f \in \mathcal{H} : \ V_\psi f \in L^1_w(G) \right\}$
Distribution space: $(\mathcal{H}^1_w)^\sim$ (anti-dual of $\mathcal{H}^1_w$)

$\Rightarrow \langle \cdot, \cdot \rangle$ extends to $(\mathcal{H}^1_w)^\sim \times \mathcal{H}^1_w$, and thus V_ψ extends from $\mathcal{H}$ to $(\mathcal{H}^1_w)^\sim$
Coorbit spaces:

$$\mathcal{C}o_\pi L^p_m(G) := \left\{ f \in (\mathcal{H}^1_w)^\sim : \ V_\psi f \in L^p_m(G) \right\} ,$$

equipped with the norm $\|f\|_{\mathcal{C}o_\pi L^p_m(G)} := \|V_\psi f\|_{L^p_m(G)}$

Test function space: $\mathcal{H}_w^1 := \left\{ f \in \mathcal{H} : \ V_\psi f \in L_w^1(G) \right\}$
Distribution space: $(\mathcal{H}_w^1)^\sim$ (anti-dual of $\mathcal{H}_w^1$)

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- $\mathcal{C}o_\pi L_m^p(G)$ is a Banach space
- $\mathcal{C}o_\pi L_w^1(G) = \mathcal{H}_w^1$
- $\mathcal{C}o_\pi L^2(G) = \mathcal{H}$
- $\mathcal{C}o_\pi L_{1/w}^\infty(G) = (\mathcal{H}_w^1)^\sim$
- $\mathcal{H}_w^1 \subseteq \mathcal{C}o_\pi L_m^p(G) \subseteq \mathcal{H}_{1/w}^\infty$

Properties on Coorbit Spaces

Proposition (Feichtinger/Gröchenig)

Let $\psi \in \mathcal{A}_w(G)$.

- $V_\psi : \mathcal{C}o_\pi L^p_m(G) \to L^p_m(G)$ *is an isometry.*
- $V^*_\psi : L^p_m(G) \to \mathcal{C}o_\pi L^p_m(G)$ *is continuous.*
- $V^*_\psi V_\psi = I_{\mathcal{C}o_\pi L^p_m(G)}$.

[6] Hans G. Feichtinger and Karlheinz Gröchenig. "Banach spaces related to integrable group representations and their atomic decompositions, I". In: J. Funct. Anal. 86.2 (1989), pp. 307–340.

Properties on Coorbit Spaces

Proposition (Feichtinger/Gröchenig, cont.)

- **Correspondence principle**: *Let* $F \in L^p_m(G)$

$$\exists f \in \mathcal{C}o_\pi L^p_m(G) \text{ s.t. } F = V_\psi f \quad \Leftrightarrow \quad F = F * V_\psi \psi$$

- **Duality**: *If* $1 \leqslant p < \infty$, *then* $(\mathcal{C}o_\pi L^p_m(G))^* = \mathcal{C}o_\pi L^q_{1/m}(G)$
- **Discretization:** $\exists \{g_i\}_{i \in \mathcal{I}} \subset G$ *and* $\lambda_i : \mathcal{C}o_\pi L^1_w(G) \to \mathbb{C}$ *s.t.*

$$f = \sum_{i \in \mathcal{I}} \lambda_i(f) \pi(g_i) \psi, \text{ with } \sum_{i \in \mathcal{I}} |\lambda_i(f)| w(g_i) \asymp \|f\|_{\mathcal{C}o_\pi L^1_w(G)}.$$

[7] Feichtinger and Gröchenig, "Banach spaces related to integrable group representations and their atomic decompositions, I".

Tensor Products

- **Simple tensor:** $\psi \otimes \phi$ formal product of two vectors $\psi, \phi \in \mathcal{H}$
- **Homogeneity:** $\alpha \cdot (\psi \otimes \phi) = (\alpha\psi) \otimes \phi = \psi \otimes (\overline{\alpha}\phi)$, $\alpha \in \mathbb{C}$
- **Tensor product:**

$$\mathcal{H} \otimes \mathcal{H} := \overline{\mathrm{span}\left(\psi \otimes \phi : \ \psi, \phi \in \mathcal{H}\right)}$$

completion w.r.t. the inner product

$$\left\langle \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \right\rangle := \left\langle \psi_1, \psi_2 \right\rangle \left\langle \phi_2, \phi_1 \right\rangle .$$

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Sometimes one can find: $\mathcal{H} \otimes \mathcal{H} := \mathcal{HS}(\mathcal{H})$ (Hilbert-Schmidt operators)

- implicitly a non-trivial kernel theorem
- we distinguish operators and tensor products

Tensor Representations

In principle: $G_1, G_2, \pi_1, \pi_2, \mathcal{H}_1, \mathcal{H}_2, \psi_1, \psi_2, m_1, m_2, w_1, w_2$

Tensor representation: $\pi_\otimes : G \times G \to \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$

$$\pi_\otimes(g,h) := \pi(h) \otimes \pi(g),$$

$\pi_\otimes$ acts on a simple tensor by

$$\pi_\otimes(g,h)(\phi \otimes \psi) = \pi(h)\phi \otimes \pi(g)\psi.$$

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Properties:

- $\pi_\otimes$ is a unitary representation of $G \times G$ on $\mathcal{H} \otimes \mathcal{H}$.
- $\pi_\otimes$ is irreducible

In case we would treat the tensor product as a space of Hilbert-Schmidt operators, $\pi_\otimes$ acts on $A \in \mathcal{HS}(\mathcal{H})$ as

$$\pi_\otimes(g, h)A = \pi(h)A\pi(g)^*.$$

Tensors and Coorbit Spaces

- Let $\Psi := \psi \otimes \psi \in \mathcal{H} \otimes \mathcal{H}$

$$V_\Psi(f_2 \otimes f_1)(g,h) = V_\psi f_2(h)\overline{V_\psi f_1(g)}.$$

$\Rightarrow \Psi$ is admissible for $\pi_\otimes$, if ψ is admissible for π

- **Separable weights:**
  - $w_\otimes(g,h) := w(g) \cdot w(h)$
  - $m_\otimes(g,h) := m(g) \cdot m_2(h)$
  - $(1/w)(g,h) := (w(g) \cdot w(h))^{-1}$

  where w is submultiplicative and m is w-moderate.

- $\Psi \in \mathcal{A}_w(G \times G)$, if $\psi \in \mathcal{A}_w(G)$, i.e. $\pi_\otimes$ is integrable $\Rightarrow$ Coorbit spaces for $\pi_\otimes$ are **well-defined**

$A : \mathcal{C}o_\pi\, L^1_w(G) \to \mathcal{C}o_\pi\, L^\infty_{1/w}(G)$
A :“test functions” $\to$ “distributions”
Goal: associate an integral operator $\mathfrak{A}$ to A

[8] Peter Balazs. “Matrix-representation of operators using frames”. In: Sampling Theory in Signal and Image Processing (STSIP) 7.1 (2008), pp. 39–54. eprint: http://arxiv.org/abs/math.FA/0510146. URL: http://arxiv.org/abs/math.FA/0510146.

[9] Peter Balazs and Karlheinz Gröchenig. “A Guide to Localized Frames and Applications to Galerkin-like Representations of Operators”. In: Frames and Other Bases in Abstract and Function Spaces. Ed. by Isaac Pesenson et al. Applied and Numerical Harmonic Analysis series (ANHA). Birkhauser/Springer, 2017. URL: https://arxiv.org/abs/1611.09692.

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Formal calculations:

$$f = \int_G \langle f, \pi(g)\psi\rangle \pi(g)\psi dg$$

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$$V_\psi(Af)(h) = \int_G \langle f, \pi(g)\psi\rangle \langle A\pi(g)\psi, \pi(h)\psi\rangle dg$$

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$$V_\psi(Af)(h) = \int_G \langle f, \pi(g)\psi\rangle k_A(g,h) dg$$

where k_A is a continuous Galerkin like representation[8,9] of A,

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Kernel Theorems

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$$k_A(g,h) = \langle A\pi(g)\psi, \pi(h)\psi\rangle.$$

Define

$$\mathfrak{A}F(h) = \int_G F(g)k_A(g,h)dg$$

$$V_\psi A = \mathfrak{A}V_\psi \quad \Leftrightarrow \quad A = V_\psi^* \mathfrak{A} V_\psi$$

Coorbit theory allows to make these manipulations precise

[8] Balazs, "Matrix-representation of operators using frames".

[9] Balazs and Gröchenig, "A Guide to Localized Frames and Applications to Galerkin-like Representations of Operators".

Kernel Theorem

Theorem (B., Gröchenig, Speckbacher)

Let π be an integrable representations of G
(i) $K \in \mathcal{C}o_{\pi_\otimes} L^\infty_{1/w}(G \times G)$ *defines* $A : \mathcal{C}o_\pi L^1_w(G) \to \mathcal{C}o_\pi L^\infty_{1/w}(G)$ *by*

$$\langle A\upsilon, \varphi \rangle = \langle K, \varphi \otimes \upsilon \rangle, \ \ \upsilon, \varphi \in \mathcal{C}o_\pi L^1_w(G).$$

for all $\upsilon, \varphi \in \mathcal{C}o_\pi L^1_w(G)$. *We have* $k_A = V_\Psi K$ *and*

$$\|A\|_{Op} \asymp \|K\|_{\mathcal{C}o_{\pi_\otimes} L^\infty_{1/w}(G \times G)},$$

(ii) ***Kernel theorem:*** *If* $A : \mathcal{C}o_\pi L^1_w(G) \to \mathcal{C}o_\pi L^\infty_{1/w}(G)$ *is bounded*

$\Rightarrow \exists\, K \in \mathcal{C}o_{\pi_\otimes} L^\infty_{1/w}(G \times G)$ *unique, s.t. the above holds.*

Proof Idea:

Ad (i): Show that $\langle K, \varphi \otimes \upsilon \rangle$ is a bdd. functional for fixed $\upsilon \Rightarrow A\upsilon$

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Ad (ii): Show that the mapping $K \mapsto A$ is unique and surjective

- **Uniqueness:**
  If $K' \neq K \in \mathcal{C}o_{\pi_{\otimes}} L^{\infty}_{1/w}(G \times G)$ is another kernel, then

$$\langle K, \varphi \otimes \upsilon \rangle = \langle K', \varphi \otimes \upsilon \rangle$$

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  If $K' \neq K \in \mathcal{C}o_{\pi_\otimes} L^\infty_{1/w}(G \times G)$ is another kernel, then

$$\sum_k \lambda_k \langle K, \varphi_k \otimes \upsilon_k \rangle = \sum_k \lambda_k \langle K', \varphi_k \otimes \upsilon_k \rangle$$

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$$\langle K, F \rangle = \langle K', F \rangle, \qquad \forall F \in \mathcal{C}o_{\pi_\otimes} L^1_w(G \times G) \quad !!!$$

Discretization Thm.:
span of simple tensors dense in $\mathcal{C}o_{\pi_\otimes} L^1_w(G \times G)$

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  Discretization Thm.:
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- **Surjectivity:**
  If $A : \mathcal{C}o_\pi L^1_w(G) \to \mathcal{C}o_\pi L^\infty_{1/w}(G)$, then $k_A \in L^\infty_{1/w}(G \times G)$
  We show: $k_A * V_\Psi \Psi = k_A$
  $\Rightarrow \exists\, K \in \mathcal{C}o_{\pi_\otimes} L^\infty_{1/w}(G \times G)$ by correspondence principle

Schur's Test

$$\|F\|_{L_m^{p,\infty}(G\times G)} := \operatorname{ess\,sup}_{h\in G}\left(\int_G |F(g,h)|^p m(g,h)^p dg\right)^{1/p},$$

$$\|F\|_{\mathcal{L}_m^{p,\infty}(G\times G)} := \operatorname{ess\,sup}_{g\in G}\left(\int_G |F(g,h)|^p m(g,h)^p dh\right)^{1/p}.$$

Proposition (Schur's Test)

Let $\frac{1}{p} + \frac{1}{q} = 1$, *and* $Tf(h) := \int_G f(g) k_T(g,h) dg$, *with* $k_T : G \times G \to \mathbb{C}$.

(*i*) $T : L^1_m(G) \to L^p_m(G)$ *bdd.* $\Leftrightarrow$ $k_T \in \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G \times G)$.

$$\|T\|_{L^1_m(G) \to L^p_m(G)} = \|k_T\|_{\mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G\times G)}.$$

(*ii*) $T : L^p_m(G) \to L^\infty_m(G)$ *bdd.* $\Leftrightarrow$ $k_T \in L^{q,\infty}_{m^{-1}\otimes m}(G \times G)$.

$$\|T\|_{L^p_m(G) \to L^\infty_m(G)} = \|k_T\|_{L^{q,\infty}_{m^{-1}\otimes m}(G\times G)}.$$

More Kernel Theorems

Theorem (B., Gröchenig, Speckbacher)

If $\frac{1}{p} + \frac{1}{q} = 1$, *&* $A : \mathcal{C}o_\pi L^1_w(G) \to \mathcal{C}o_\pi L^\infty_w(G)$ *bdd. with kernel K, then:*

(i) $A : \mathcal{C}o_\pi L^1_m(G) \to \mathcal{C}o_\pi L^p_m(G)$ *bdd.* $\Leftrightarrow K \in \mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G \times G)$.

$$\|A\|_{Op} \asymp \|K\|_{\mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}_{m_1^{-1}\otimes m_2}(G)}.$$

(ii) $A : \mathcal{C}o_\pi L^p_m(G) \to \mathcal{C}o_\pi L^\infty_m(G)$ *bdd.* $\Leftrightarrow K \in \mathcal{C}o_{\pi_\otimes} L^{q,\infty}_{m^{-1}\times m}(G \times G)$.

$$\|A\|_{Op} \asymp \|K\|_{\mathcal{C}o_{\pi_\otimes} L^{q,\infty}_{m_1^{-1}\otimes m_2}(G)}.$$

[10] Balazs, Gröchenig, and Speckbacher, "Kernel theorems in coorbit theory".

Proof Sketch:

Consider (i):

- Assumptions guarantee: $\exists\, K$ and $V_\Psi K = k_A$
- Formal calculations: $A = V_\psi^* \mathfrak{A} V_\psi$
- $V_\psi : \mathcal{C}o_\pi L_m^1(G) \to L_m^1(G)$ isometry
- $V_\psi^* : L_m^p(G) \to \mathcal{C}o_\pi L_m^p(G)$ bounded

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$''\Leftarrow''$: If $K \in \mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G \times G)$, then $A : \mathcal{C}o_\pi L^1_m(G) \to \mathcal{C}o_\pi L^p_m(G)$ is bounded, since

$$\|A\|_{Op} \leqslant \|V_\psi^*\|_{Op} \, \|\mathfrak{A}\|_{L^1_m(G) \to L^p_m(G)} \, \|V_\psi\|_{Op}$$
$$\leqslant C \|k_A\|_{\mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G\times G)} = C\|K\|_{\mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G\times G)}.$$

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"$\Rightarrow$": Assume $A : \mathcal{C}o_\pi\, L^1_m(G) \to \mathcal{C}o_\pi\, L^p_m(G)$ is bounded, then

$$\begin{aligned}
\|K\|_{\mathcal{C}o_\pi\, \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G\times G)} &= \|V_\Psi K\|_{\mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G\times G)} \\
&= \sup_{g\in G} \left(\int_G |\langle A\pi(g)\psi, \pi(h)\psi\rangle m(h)|^p dh \right)^{1/p} m(g)^{-1} \\
&= \sup_{g\in G} \|A\pi(g)\psi\|_{\mathcal{C}o_\pi\, L^p_m(G)} m(g)^{-1}. \\
&\leqslant \|A\|_{Op} \sup_{g\in G} \|\pi(g)\psi\|_{\mathcal{C}o_\pi\, L^1_m(G)} m(g)^{-1} \leqslant C\|A\|_{Op}
\end{aligned}$$

Kernel Theorems

Corollary

The following conditions are equivalent:

- $A : \mathcal{C}o_\pi L^p_m(G) \to \mathcal{C}o_\pi L^p_m(G)$ *is bounded for* **every** $1 \leqslant p \leqslant \infty$.
- $A : \mathcal{C}o_\pi L^1_m(G) \to \mathcal{C}o_\pi L^1_m(G)$ **and**

  $A : \mathcal{C}o_\pi L^\infty_m(G) \to \mathcal{C}o_\pi L^\infty_m(G)$ *are bounded.*
- $K \in \mathcal{C}o_{\pi_\otimes} \mathcal{L}^{1,\infty}_{m^{-1}\otimes m}(G \times G) \bigcap \mathcal{C}o_{\pi_\otimes} L^{1,\infty}_{m^{-1}\otimes m}(G \times G)$.

Schematic Diagram

$$
\begin{array}{ccc}
\mathcal{C}o_{\pi} L^1_m(G) & \xrightarrow[\textstyle A \text{ bounded}]{} & \mathcal{C}o_{\pi} L^p_m(G) \\
 & \updownarrow & \\
\Big\downarrow V_\psi & K \in \mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G \times G) & \Big\downarrow V_\psi \\
 & \Big\downarrow V_\Psi & \\
 & k_A \in \mathcal{L}^{p,\infty}_{m^{-1}\otimes m}(G \times G) & \\
L^1_m(G) & \xrightarrow[\textstyle \mathfrak{A} \text{ bounded}]{} & L^p_m(G)
\end{array}
$$

Sufficient Conditions

- So far: Correspondence between coorbit spaces of kernels and boundedness of operators
- Sufficient versions of Schur's test: conditions for e.g. regularizing operators

Theorem

If the unique kernel of A satisfies $K \in \mathcal{C}o_{\pi_\otimes} L^1_w(G \times G)$, then A is bounded from $\mathcal{C}o_\pi L^\infty_{1/w}(G)$ to $\mathcal{C}o_\pi L^1_w(G)$.

- Feichtinger & Jakobsen: Full characterization of operators with kernel in $\mathcal{C}o_{\pi_\otimes} L^1(G \times G)$ in the case of **modulation spaces**[11]

[11] Mads S. Jakobsen and Hans G. Feichtinger. "The inner kernel theorem for a certain Segal algebra". 2018. URL: https://arxiv.org/abs/1806.06307.

Examples - Modulation spaces

- The Weyl-Heisenberg group $G = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ is given by

$$(x, \omega, e^{2\pi i\tau}) \cdot (x', \omega', e^{2\pi i\tau'}) = (x + x', \omega + \omega', e^{2\pi i(\tau+\tau'-x\cdot\omega')}).$$

- Translation: $T_x f(t) = f(t-x)$, Modulation: $M_\omega f(t) = e^{2\pi i\omega t} f(t)$
- Projective representation: $\pi(z) = \pi(x, \omega) = M_\omega T_x, \;\; z = (x, \omega)$
- Short-time Fourier transform: $V_\psi f(x, \omega) = \langle f, M_\omega T_x \psi \rangle$

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- Short-time Fourier transform: $V_\psi f(x, \omega) = \langle f, M_\omega T_x \psi \rangle$
- Coorbit spaces w.r.t. to π are **modulation spaces**:

$$\mathcal{C}o_\pi L^p(\mathbb{R}^d) = M^p(\mathbb{R}^d)$$

Kernel Theorem

For every bounded operator $A : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$, there exists a kernel $K \in \mathcal{C}o_{\pi_\otimes} L^\infty(G \times G)$, s.t. $\langle A\upsilon, \varphi \rangle = \langle K, \varphi \otimes \upsilon \rangle$, $\upsilon, \varphi \in M^1(\mathbb{R}^d)$.

Examples - Modulation spaces

- $\pi_{\otimes}(z_1, z_2)$ is just the TF-shift $M_{(\omega_1, -\omega_2)} T_{(x_1, x_2)}$ on $L^2(\mathbb{R}^{2d})$:

$$\pi_{\otimes}(z_1, z_2)\Psi = (M_{\omega_1} T_{x_1} \psi) \otimes (M_{\omega_2} T_{x_2} \psi) = M_{(\omega_1, -\omega_2)} T_{(x_1, x_2)} \Psi$$

- Coorbit spaces for $G \times G$ w.r.t. $\pi_{\otimes}$ are modulation spaces on $\mathbb{R}^{2d}$

$$\|K\|_{M^{\infty}(\mathbb{R}^{2d})} = \|K\|_{\mathcal{C}o_{\pi_{\otimes}} L^{\infty}(\mathbb{R}^{4d})}$$

[12] Cordero and Nicola, "Kernel theorems for modulation spaces".

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- Recover Feichtinger's kernel theorem:

For every $A : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ bounded, there exists a unique kernel $K \in M^\infty(\mathbb{R}^{2d})$, s.t. $\langle Af, g \rangle = \langle K, \varphi \otimes \upsilon \rangle$, $\upsilon, \varphi \in M^1(\mathbb{R}^d)$.

- Recover kernel theorems by Cordero & Nicola[12] :
  Interpret the mixed-norm spaces as mixed modulation spaces

$$\begin{array}{lll} (i)\ A : M^1(\mathbb{R}^d) \to M^p(\mathbb{R}^d) & \text{bdd.} \Leftrightarrow & K \in \mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}(\mathbb{R}^{4d}) \\ (ii)\ A : M^p(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d) & \text{bdd.} \Leftrightarrow & K \in \mathcal{C}o_{\pi_\otimes} L^{q,\infty}(\mathbb{R}^{4d}) \end{array}$$

[12] Cordero and Nicola, "Kernel theorems for modulation spaces".

Examples - Besov spaces

- Affine group $G = \mathbb{R} \times \mathbb{R}^*$ given by $(x,a) \cdot (y,b) = (x + ay, ab)$
- Dilation: $D_a f(t) = a^{-1/2} f(t/a)$, Representation: $\pi(x,a) = T_x D_a$
- Continuous wavelet transform: $W_\psi f(x,a) := \langle f, \pi(x,a)\psi \rangle$
- $\mathcal{C}o_\pi L^p(G) = \dot{B}_{p,p}^{-1/2+1/p}(\mathbb{R})$ homogeneous Besov spaces

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Kernel Theorem

For every $A : \dot{B}^{1/2}_{1,1}(\mathbb{R}) \to \dot{B}^{-1/2}_{\infty,\infty}(\mathbb{R})$ bounded there exists a unique kernel $K \in \mathcal{C}o_{\pi_\otimes} L^\infty(G \times G)$, s.t. $\langle A\upsilon, \varphi \rangle = \langle K, \varphi \otimes \upsilon \rangle$, $\upsilon, \varphi \in \dot{B}^{1/2}_{1,1}(\mathbb{R})$.

- $\mathcal{C}o_{\pi_\otimes} L^\infty(G \times G) = S^{-1/2,-1/2}_{\infty,\infty} B(\mathbb{R}^2)$ Besov space of dominating mixed smoothness[13]

ÖAW ÖSTERREICHISCHE AKADEMIE DER WISSENSCHAFTEN

Examples - Besov spaces

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(i) $A : \dot{B}^{1/2}_{1,1}(\mathbb{R}) \to \dot{B}^{-1/2+1/p}_{p,p}(\mathbb{R})$ bdd $\Leftrightarrow$ $K \in \mathcal{C}o_{\pi_\otimes} \mathcal{L}^{p,\infty}(G \times G)$

(ii) $A : \dot{B}^{-1/2+1/p}_{p,p}(\mathbb{R}) \to \dot{B}^{-1/2}_{\infty,\infty}(\mathbb{R})$ bdd $\Leftrightarrow$ $K \in \mathcal{C}o_{\pi_\otimes} L^{q,\infty}(G \times G)$

Summary

- **Step 1:** Show the existence of kernel K for operator A
- **Step 2:**
  - Associate an integral operator $\mathfrak{A}$ with equivalent norm to the operator A
  - Identify the integral kernel of $\mathfrak{A}$ as $V_\Psi K$
  - Use Schur's test to get wide range of kernel theorems
- These "finer" characterizations do not have a counterpart in distribution theory
- Operators on well-known spaces can be characterized
- Operators between coorbit spaces of different groups can be characterized

True for localized frames!

- Cordero and Nicola argued that "this reveals the superiority, in some respects, of the modulation space formalism upon distribution theory"
- We added a more abstract point of view and argued that the deeper reason for this superiority lies in the theory of coorbit spaces and in the convenience of Schur's test for integral operators.
- In the future we will argue that the group structure is not necessary.

Thank you!

Questions? Comments?