On a recent characterization of Riesz bases

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Dedicated to the memory of Professor Arpad Takači

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Overview of the talk

1. Notation and Basic Definitions
2. Recall Riesz bases - def., classical characterizations
3. Focus on one of the characterizations (the one via complete biorthogonal sequences)
4. Main result - Progress on this characterization
5. Applications

Notation and basic Definitions

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Def. F - Bessel sequence in $\mathcal{H}$ if $\exists B \in (0, \infty)$:

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Def. F - minimal if $f_n \notin \overline{\text{span}}(f_k)_{k \neq n}^{\infty}, \ \forall n \in \mathbb{N}$.

Def. Riesz basis for $\mathcal{H}$ - sequence of the form

$$(Ve_k)_{k=1}^{\infty}$$

with $V : \mathcal{H} \to \mathcal{H}$ - bounded bijective op. on $\mathcal{H}$.

Ex

Recall Riesz bases

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Riesz bases - introduced by Nina Bari (1946, 1951)

Recall: The following are equivalent:

($\mathcal{R}_1$) $(f_k)_{k=1}^{\infty}$ forms a Riesz basis for $\mathcal{H}$.

($\mathcal{R}_2$) $(f_k)_{k=1}^{\infty}$ is a bounded unconditional basis for $\mathcal{H}$.

($\mathcal{R}_3$) $(f_k)_{k=1}^{\infty}$ is a basis for $\mathcal{H}$ such that $\sum_{k=1}^{\infty} c_k f_k$ converges in $\mathcal{H}$ $\Leftrightarrow$ $\sum_{k=1}^{\infty} |c_k|^2 < \infty$.

($\mathcal{R}_4$) $(f_k)_{k=1}^{\infty}$ is complete in $\mathcal{H}$ and its Gram matrix $(\langle f_k, f_j \rangle)_{j,k=1}^{\infty}$ determines a bounded bijective operator on ℓ^2.

(continuation)

$(\mathcal{R}_5)$ $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$ and $\exists A, B \in (0, \infty)$:

$$A \sum |c_k|^2 \le \| \sum c_k f_k \|^2 \le B \sum |c_k|^2 \qquad (1)$$

$\forall$ finite (c_k) (hence $\forall$ $(c_k)_{k=1}^\infty \in \ell^2$).

$(\mathcal{R}_6)$ $(f_k)_{k=1}^\infty$ is a complete Bessel sequence in $\mathcal{H}$ and it has a biorthogonal sequence $(g_k)_{k=1}^\infty$: $(g_k)_{k=1}^\infty$ is also a complete Bessel sequence in $\mathcal{H}$.

$(\mathcal{R}_7)$ $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$ and it has a complete biorthogonal $(g_k)_{k=1}^\infty$: $\sum_{k=1}^\infty |\langle f, f_k \rangle|^2 < \infty$ and $\sum_{k=1}^\infty |\langle f, g_k \rangle|^2 < \infty \quad \forall f \in \mathcal{H}$.

(continuation)

$(\mathcal{R}_5)$ $(f_k)_{k=1}^{\infty}$ is complete in $\mathcal{H}$ and $\exists A, B \in (0, \infty)$:

$$A \sum |c_k|^2 \leq \| \sum c_k f_k \|^2 \leq B \sum |c_k|^2 \qquad (2)$$

$\forall$ finite (c_k) (hence $\forall$ $(c_k)_{k=1}^{\infty} \in \ell^2$).

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($\mathcal{R}_5$) $(f_k)_{k=1}^{\infty}$ is complete in $\mathcal{H}$ and $\exists A, B \in (0, \infty)$:

$$A \sum |c_k|^2 \leq \| \sum c_k f_k \|^2 \leq B \sum |c_k|^2 \qquad (3)$$

$\forall$ finite (c_k) (hence $\forall$ $(c_k)_{k=1}^{\infty} \in \ell^2$).

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On $\mathcal{R}_5$ and $\mathcal{R}_6$:

The lower condition in (3) - more difficult to check than the upper one!

Completeness - less restrictive, but not easy to check either!

Main Result:

Removal of

a completeness-condition in $\mathcal{R}_6$

MAIN RESULT

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$\Leftrightarrow$

$(\mathcal{R}_6^{new1})$ $(f_k)_{k=1}^{\infty}$ is a ~~complete~~ Bessel sequence in $\mathcal{H}$ and it has a biorthogonal sequence $(g_k)_{k=1}^{\infty}$: $(g_k)_{k=1}^{\infty}$ is also a complete Bessel sequence in $\mathcal{H}$.

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$(\mathcal{R}_6^{new2})$ $(f_k)_{k=1}^{\infty}$ is a complete Bessel sequence in $\mathcal{H}$ and it has a biorthogonal sequence $(g_k)_{k=1}^{\infty}$: $(g_k)_{k=1}^{\infty}$ is also a ~~complete~~ Bessel sequence in $\mathcal{H}$.

- It reduces $(\mathcal{R}_6)$ to a characterization which does not contain extra-conditions.

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Importance and Applications of $\mathcal{R}_6^{new}$

- It reduces $(\mathcal{R}_6)$ to a characterization which does not contain extra-conditions.
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- It leads to conclusions which can not be delivered from

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E.g., consider:
$g(x) = e^{-\pi x^2}$
Λ - sequence of uniformly separated points from $\mathbb{R}^2$
the Gabor system $G_{g,\Lambda} = (e^{2\pi i \mu \cdot} g(\cdot - \tau))_{(\tau,\mu)\in\Lambda}$.

Known:

- $G_{g,\Lambda}$ is Not a Riesz basis for $L^2(\mathbb{R})$!
- $G_{g,\Lambda}$ is a Bessel sequence!
- $G_{g,\Lambda}$ is minimal and complete for certain Λ! ex

The new result $\Rightarrow$

the biorthogonal of $G_{g,\Lambda}$ is not Bessel! see

Further applications expected.

THANK YOU

FOR YOUR ATTENTION!

THANKS

TO THE CONFERENCE ORGANIZERS!