Multipliers: from Fourier to continuous frames

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- part I
  towards an infinite dimensional setting
- part II
  Fourier multipliers in a broad context, with the moral:
  seemingly different problems may have hidden connections
- part III
  continuous frames and time-frequency localization

part I

- Matrices are linear transforms:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

- If $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq 0$, then λ is the eigenvalue and $\mathbf{x}$ is the eigenvector of A.
- Google PageRank attempts to return the best ranking of websites when searching on the web. The goal of the algorithm is to find the eigenvector of a certain Markov transition matrix which corresponds to the largest eigenvalue (in this case $\lambda = 1$).[1]

[1] How Google works: Markov chains and eigenvalues, http://blog.kleinproject.org/?p=280

- ”Eigenvalues quantify the importance of information along the line of eigenvectors. Equipped with this information, we know what part of the information can be ignored and how to compress information....
  It also helps us to extract features in developing machine learning models. Sometimes, it makes the model easier to train because of the reduction of tangled information.”[2]
- The size of matrices involved in PageRank[3] and machine learning applications indicates to take into consideration appropriate counterparts in infinite dimensional setting.

[2] Machine Learning and Linear Algebra – Eigenvalue and eigenvector, by J. Hui, February 20, 2019, https://medium.com/

[3] e.g. $30 \cdot 10^9$

- Consider the system of linear equations:

$$
\begin{aligned}
1 &= x_1 + x_2 + x_3 + \cdots + x_n \\
1 &= x_2 + x_3 + \cdots + x_n \\
1 &= x_3 + \cdots + x_n \\
&\dots \\
&\dots \\
1 &= x_n
\end{aligned}
$$

- Its solution is $(0, 0, \dots, 1)$.

- Consider the infinite system of linear equations:

$$
\begin{aligned}
1 &= x_1 + x_2 + x_3 + \cdots + x_n + x_{n+1} + \ldots \\
1 &= x_2 + x_3 + \cdots + x_n + x_{n+1} + \ldots \\
1 &= x_3 + \cdots + x_n + x_{n+1} + \ldots \\
& \ldots \\
& \ldots \\
1 &= x_n + x_{n+1} + \ldots \\
1 &= x_{n+1} + \ldots \\
& \ldots \\
& \ldots
\end{aligned}
$$

- There is no solution at all.

- Compact operators are a natural generalization of finite-rank operators in an infinite-dimensional setting.
- A finite-rank operator is of a particular form:

$$Ax = \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle f_k, \quad x \in \mathcal{H},$$

for some $\lambda_k \geq 0$, where $\{e_k\}$ and $\{f_k\}$ are ONB of the Hilbert space $\mathcal{H}$.
- Remark: If $n = \infty$ and if $\lambda_k \searrow 0$, when k tends to infinity, then A is a compact operator. If, moreover $\sum_{k=1}^{\infty} \lambda_k < \infty$, then A is a trace class compact operator. (These are not definition, but sufficient conditions.)
- Compact operators historically originated from integral operators. In fact, Fredholm[4] intended to find solutions of integral equations by solving appropriate systems of linear equations.

[4] Ivar Fredholm (1866 – 1927), Swedish mathematician

SUR UNE CLASSE D'ÉQUATIONS FONCTIONNELLES

PAR

IVAR FREDHOLM

À STOCKHOLM.

Dans quelques travaux [1] ABEL s'est occupé avec le problème de déterminer une fonction $\varphi(x)$ de manière qu'elle satisfasse à l'équation fonctionnelle

(a) $$\int f(x, y)\varphi(y)dy = \psi(x)$$

$f(x, y)$ et $\psi(x)$ étant des fonctions données. ABEL a résolu quelques cas particuliers de cette équation fonctionnelle dont il paraît avoir reconnu le premier l'importance. C'est pour cela que je propose d'appeler l'équation fonctionnelle (a) une *équation fonctionnelle abélienne.*

Dans cette note je ne m'occupe pas en premier lieu de l'équation abélienne mais de l'équation fonctionnelle

(b) $$\varphi(x) + \int_0^1 f(x, y)\varphi(y)dy = \psi(x),$$

qui est étroitement liée à l'équation abélienne.

En effet, si on introduit au lieu de $f(x, y)$ et $\psi(x)$, $\frac{1}{\lambda} f(x, y)$ et $\frac{1}{\lambda}\psi(x)$, l'équation (b) s'écrit

(c) $$\lambda\varphi(x) + \int_0^1 f(x, y)\varphi(y)dy = \psi(x),$$

équation qui se transforme en l'équation (a) en posant $\lambda = 0$. Ainsi la solution de l'équation (a) peut être considérée comme implicitement contenue dans la solution de l'équation (b).

[1] Magazin for Naturvidenskaberne, Kristiania 1823 et Oeuvres complètes.

Acta mathematica. 27. Imprimé le 30 mars 1903.

- It seems that Hilbert[5] expected that it would be possible to prove Riemann hypothesis by using Fredholm's technique.[6]
- Hilbert published 6 papers in 1904–1910 "These papers are among the most influential papers written in the 20th century."[7]
- Among other things, Hilbert introduced there calculations which include scalar products between functions.
- In 1913 Riesz[8] used the words "Hilbert space " to describe such structure.

[5] David Hilbert (1862 – 1943), German mathematician

[6] cf. B. Simon, Operator Theory, A Comprehensive Course in Analysis, Part 5, AMS, 2015.

[7] N. L. Carothers

[8] Riesz Frigyes (1880 – 1956), Hungarian mathematician

- In quantum mechanics, physical quantities (position, momentum, energy) are represented by operators on a certain Hilbert space.
- "Eigenfunctions (which belong the Hilbert space) of some dynamical variable (i.e. operator acting on that space) are those states of the physical system for which that particular dynamical variable has what is called "a definite value", and the eigenvalues is the actual "value" that dynamical variable has for that state."[9]

[9] R. Penrose, The Road to Reality, 2007.

conclusion of part I

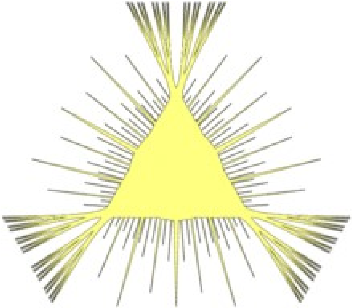
- Compact operators (acting on Hilbert space), and trace class operators in particular, are infinite dimensional counterparts of systems of linear equations. Some concepts from finite dimensional linear algebra (such as scalar product, eigenvalue, eigenvector, trace) have an infinite dimensional interpretations, and even physical meaning.
- Therefore, when considering localization operators in time-frequency analysis, apart from continuity (boundedness), it is of interest to study their compactness properties.
- We are ready for the definition.

- A linear operator A from the Hilbert space $\mathcal{H}_1$ into the Hilbert space $\mathcal{H}_2$ is called compact if the image of any bounded sequence contains a convergent subsequence.
- For any compact operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, the operator $A^*A : \mathcal{H}_1 \to \mathcal{H}_1$ is compact and non-negative.[10] The unique operator S such that $S^2 = A^*A$, is also compact.
- The eigenvalues of S are called the singular values of A. The sequence of singular values (λ_n) converges to zero, or consists of only finitely many nonzero terms.
- The operator A is trace class operator if $\sum_n |\lambda_n| < \infty$.

[10] $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ is given by $\langle Ax, y\rangle = \langle x, A^*y\rangle$, $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$

part II

- Next we mention 3 conjectures related to seemingly different problems, but apparently intimately related.
- We begin with the Kakeya conjecture.
- In 1917, Sōichi Kakeya asked for a set of minimal area in the plane through which we can rotate a unit length needle by $360°$.
- Independently, Besikovitch (1919) constructed sets with zero Lebesgue measure which contains a unit line segment in every direction.
- If we require that the set contains a unit line segment which can be rotated continuously in every direction within the set, then such set can have arbitrary small measure, but never measure zero (Cunningham, 1971).



- However, if Hausdorff dimension is considered instead of the Lebesgue measure, then the Kakeya conjecture can be stated as follows.[11]
- If E is a Kakeya set in $\mathbb{R}^d$, then the Hausdorff dimension of the set E, $\dim E$ is equal to d.
- For $d = 2$ the affirmative answer is given by Davis, 1971.
- For $d \geq 3$, since 1991, some answers are given by Bourgain, Wolff, Katz, Tao, ... For example, it is proved that $\dim E \geq \frac{d}{2} + 1$.
- "This combinatorial problem plays a major role in the theory of oscillatory integrals in harmonic analysis."[12]

[11] Recall, the famous Cantor set is of the Lebegsue measure zero, but it's Hausdorff dimension is $\log_3 2$.

[12] J. Bourgain, in Mathematics: Frontiers and Perspectives, IMU, AMS, 2000

- Let us meet an oscillatory integral.
- Let f be an absolutely integrable function. Then its Fourier transform is given by

$$\mathcal{F}(f(\cdot))(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \xi x}\, dx, \quad \xi \in \mathbb{R}^d.$$

  We may try to understand in what sense the inversion property

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi = \mathcal{F}^{-1}(\hat{f}(\cdot))(x), \quad x \in \mathbb{R}^d,$$

  holds when f is a function on $\mathbb{R}^d$, since $\hat{f}(\xi)$ may not be integrable.
- A choice of summability method is to introduce a localizing factor m such that $m(0) = 1$ and which decays sufficiently rapidly at infinity, and observe

$$\lim_{R\to\infty} \int_{\mathbb{R}^d} m(\frac{\xi}{R})\hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi, \quad x \in \mathbb{R}^d.$$

- For example, m could be $\chi_{B(0,1)}$, characteristic function of the unit ball.
- A function m is called an $L^p(\mathbb{R}^d)$ Fourier multiplier if the mapping

$$f(x) \mapsto \int_{\mathbb{R}^d} m(\xi)\hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi = \mathcal{F}^{-1}(m(\cdot)\mathcal{F}f(\cdot))(x), \quad x \in \mathbb{R}^d,$$

is bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

- Here, $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, if

$$\int_{\mathbb{R}^d} |f(x)|^p\, dx < \infty.$$

In particular, $L^2(\mathbb{R}^d)$ is Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\mathbb{R}^d).$$

- If the multiplier m has enough derivatives, and is of certain polynomial decay at infinity,

$$|x|^{|\alpha|}|D^\alpha m| \leq C \quad 0 \leq |\alpha| \leq d+2,$$

  then m is an L^p Fourier multiplier for all $1 < p < \infty$.[13]
- For $d \geq 2$, the map

$$f(x) \mapsto \int_{|\xi|\leq 1} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi, \quad x \in \mathbb{R}^d,$$

  is bounded on $L^p(\mathbb{R}^d)$ if and only if $p = 2$. (Fefferman, 1971)
  A key ingredient in argument is the existence of measure-zero Kakeya sets!

[13] This is a special case of the Hörmander–Mikhlin theorem.

- An interesting case arises when the multiplier is given by $m_\lambda(\xi) = (1 - |\xi|^2)_+^\lambda$, $\lambda \geq 0$. Notice that $m_\lambda(\xi/R) \to 1$, as $R \to \infty$, so it fits within the summability method.
- Consider the Bochner–Riesz means of order $\lambda > 0$:

$$B_R^\lambda(f)(x) = \int_{\mathbb{R}^d} m_\lambda(\frac{\xi}{R})\, \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi, \quad x \in \mathbb{R}^d,\ R > 0.$$

- The Bochner–Riesz conjecture states that $m_\lambda(\xi)$ acts as a bounded Fourier multiplier on $L^p(\mathbb{R}^d)$, $p \neq 2$, if and only if $\lambda > 0$ and

$$\frac{2d}{d+1+2\lambda} \leq p \leq \frac{2d}{d-1-2\lambda}.$$

- $d = 2$ is proved by Carleson–Sjölin in 1972. Other proofs and extensions: Hörmander, Cordoba, Fefferman, Bourgain, Wolff...
- The main combinatorial component in Bourgain's approach (1991) is based on the knowledge of Kakeya-type problems.

- Another question concerning the Fourier transform is related to its restriction to subsets S of $\mathbb{R}^d$.
- If $f \in L^1(\mathbb{R}^d)$ then $\hat{f}$ is a continuous function which vanishes at infinity, so it can be restricted to any $S \subset \mathbb{R}^d$.
- If $f \in L^2(\mathbb{R}^d)$ then $\hat{f} \in L^2(\mathbb{R}^d)$, and there is no meaningful way to restrict it to any set S of zero measure.
- When $1 < p < 2$, we can, for instance, consider $f(x) = (1 + |x_1|)^{-1}$, $d \geq 2$, which belongs to $L^p(\mathbb{R}^d)$, but its Fourier transform is infinite on every point on the hyperplane $\{\xi \in \mathbb{R}^d \mid \xi_1 = 0\}$.
- In 1967 Stein made the surprising discovery that when S is of measure zero, but it contains sufficient "curvature", than one can indeed restrict the Fourier transform of $L^p(\mathbb{R}^d)$ functions for certain $p > 1$.

- The restriction problem thus can be formulated as follows: for which sets $S \subset \mathbb{R}^d$ and which $1 \leq p \leq 2$ can the Fourier transform of $f \in L^p(\mathbb{R}^d)$ be meaningfully restricted?
- Classical results in that direction have played an essential role in the Cauchy theory of dispersive PDE's.
- The restriction conjecture in one of its forms states that

$$\|\hat{f}|_S\|_{L^1(d\sigma)} \leq C_p \|f\|_p, \quad \text{for} \quad p < \frac{2d}{d+1},$$

where $d\sigma$ denotes the surface measure of the unit sphere.
- "The difficulty of the Bochner-Riesz and restriction conjectures is seen by their connection to the Kakeya conjecture."[14]

[14] B. Simon, Harmonic Analysis, A Comprehensive Course in Analysis, Part 3, AMS, 2015.

conclusion of part II

- Kakeya problem is related to combinatorial geometry, while Bochner–Riesz conjecture is a statement about oscillatory integrals, and restriction problems are related to the Cauchy theory of dispersive PDE's.
- Bochner–Riesz conjecture $\Longrightarrow$ Restriction conjecture $\Longrightarrow$ Kakeya conjecture
- There are some implications in other directions, and any progress in one of the fields makes an impact to the others.
- the moral: seemingly different problems may manifest interesting connections.

part III

- In different contexts signal analysis it is of interest to treat the time-frequency plane as one geometric whole (phase space) rather than to consider tima and frequency in separate ways.
  Consequently, analogues of Fourier multipliers are operators which localize in both time and frequency in phase-space.
- Such operators were introduced by Berezin around 1970, and applied to quantization problems in quantum mechanics.
- In signal analysis they are related to localization technique developed by Slepian-Polak-Landau around 1960.
- Basic facts on time-frequency localization operators with references to applications in optics and signal analysis are given in

I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, 1988.

When the symbol (multiplier) of Daubechies' operator is an absolutely integrable radial function, then is eigenfunctions Hermite functions, and eigenvectors are explicitly calculated.

- Localization operators of the form

$$\langle L_{\chi_\Omega} f, g\rangle = \iint_\Omega W(f,g),$$

where

$$W(f,g)(x,\omega) = \int f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega t}\,dt,\;\; f,g\in L^2(\mathbb{R}),$$

is the (cross-)Wigner distribution were studied in

J. Ramanathan, P. Topiwala, Time-frequency localization via the Weyl correspondence. *SIAM J. Math. Anal.*, 24(5): 1378-1393, 1993.

For any open set $\Omega\subset[-B,B]\times[-T,T]$ s.t. all its cross-sections in both ξ and x directions consist of at most M intervals, eigenfunctions of L_{χ_Ω} belong to $\mathcal{S}^{(1)}(\mathbb{R}^d)$.

- Recall, $f\in\mathcal{S}^{(1)}(\mathbb{R}^d)$ if and only if

$$\sup_{x\in\mathbb{R}^d}|f(x)|e^{h\cdot|x|}<\infty \quad\text{and}\quad \sup_{\omega\in\mathbb{R}^d}|\hat{f}(\omega)|e^{h\cdot|\omega|}<\infty,\quad \forall h>0.$$

- Inverse problem for a simply connected localization domain is studied in

  L. D. Abreu, M. Dörfler, An inverse problem for localization operators, *Inverse Problems* 28 (11), 115001, 16 pp, 2012.

  If one of the eigenfunctions of Daubechies' operator is a Hermite function, then the domain is a disc centered at 0.

- Localization operators on a locally compact group G and $L^p(G)$, $1 \leq p \leq \infty$, were studied in

  M. W. Wong, Wavelet transforms and localization operators, Birkhäuser Verlag, Basel, 2002.

  There one can find a product formula and trace-class properties of localization operators.

- Since the beginning of the XXI century, localization operators in the context of modulation spaces were studied by many authors:

- H. G. Feichtinger, K. Nowak, A first survey of Gabor multipliers, in Advances in Gabor Analysis, Birkhäuser, 99–128, 2003.
- E. Cordero, K. Gröchenig, Time-frequency analysis of localization operators, *J. Funct. Anal.* 205 (1), 107-131, 2003.
- Á. Bényi, K. A. Okoudjou, Bilinear pseudodifferential operators on modulation spaces. *J. Fourier Anal. Appl.* 10 (3), 301-313, 2004.
- Á. Bényi, K. H. Gröchenig, C. Heil, K. A. Okoudjou, Modulation spaces and a class of bounded multilinear pseudodifferential operators. *J. Operator Theory* 54 (2), 387-399, 2005.
- J. Toft, Continuity and Schatten properties for Toeplitz operators on modulation spaces, in *Oper. Theory Adv. Appl., 172*, 313-328, 172, Birkhäuser, 2007.
- E. Cordero, K. A. Okoudjou, Multilinear localization operators, *J. Math. Anal. Appl.* 325 (2), 1103-1116, 2007.

- The short-time Fourier transform (STFT) of $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ w.r.t. $g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus 0$ is given by:

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt, = \langle f, M_\omega T_x g \rangle,$$

  and extends to $f \in \mathcal{S}^{(1)\prime}(\mathbb{R}^d)$ by duality.
  Here T and M denote translation and modulation operators:

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_x f(\cdot) = e^{2\pi i x \cdot} f(\cdot), \quad x \in \mathbb{R}^d.$$

- The STFT inversion formula for $f, \varphi_1, \varphi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ reads as follows:

$$f(t) = \frac{1}{\langle \varphi_2, \varphi_1 \rangle} \int_{\mathbb{R}^{2d}} V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega, \quad t \in \mathbb{R}^d.$$

- Localization operator A_a^φ with symbol $a \in \mathcal{S}^{(1)\prime}(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$:

$$A_a^\varphi f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega.$$

- A natural analogue of $L^p(\mathbb{R}^d)$ spaces in the context of the STFT are Feichtinger's modulation spaces.
  Let $\phi \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus 0$, $s, t \in \mathbb{R}$ and $p, q \in [1, \infty]$. The *modulation space* $M^{p,q}_{s,t}(\mathbb{R}^d)$ consists of all $f \in (\mathcal{S}^{(1)})'(\mathbb{R}^d)$ s. t.

$$\|f\|_{M^{p,q}_{s,t}} \equiv \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi f(x,\omega) \langle x \rangle^t \langle \omega \rangle^s|^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty$$

  (with obvious interpretation of the integrals when $p = \infty$ or $q = \infty$).
  We use the usual abbreviations: $M^{p,p}_{0,0} = M^p$, $M^{p,p}_{t,t} = M^p_t$, etc.
- The most important results in the context of localization operators on modulation spaces are given in

E. Cordero, K. Gröchenig, Time-frequency analysis of localization operators, *J. Funct. Anal.* 205 (1), 107-131, 2003.

- For example, among other things, they shown the following continuity property:

$$\|A^\varphi_a\|_{op} \lesssim \|a\|_{M^\infty_{-s,0}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^1_s}.$$

- Time-frequency localization operators, also known as the STFT multilpiers, can be considered as a particular case of multipliers for continuous frames introduced in

Balazs, P., Bayer, D. and Rahimi, A.: *Multipliers for continuous frames in Hilbert spaces*, Journal of Physcis A: Mathematical and Theoretical, **45**, 2012.

- Let $\mathcal{H}$ be a complex Hilbert space and (X, μ) be a measure space with positive measure μ. The mapping $F : X \to \mathcal{H}$ is called a continuous frame with respect to (X, μ), if

  1. F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, $x \to \langle f, F(x) \rangle$ is a measurable function on X;
  2. there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \int_X |\langle f, F(x) \rangle|^2 \, d\mu(x) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1}$$

The constants A and B are called continuous frame bounds.

- Let F and G be continuous frames for $\mathcal{H}$ with respect to (X, μ) and $m : X \to \mathbb{C}$ be a measurable function. The operator $\mathbf{M}_{m,F,G} : \mathcal{H} \to \mathcal{H}$ weakly defined by

$$\langle \mathbf{M}_{m,F,G} f, g \rangle = \int_X m(x) \langle f, F(x) \rangle \langle G(x), g \rangle d\mu(x), \quad f, g \in \mathcal{H},$$

is called continuous frame multiplier of F and G with respect to the symbol m.
We use the following notation to be understood in weak sense as above:

$$\mathbf{M}_{m,F,G} f := \int_X m(x) \langle f, F(x) \rangle G(x) d\mu(x).$$

The operator $\mathbf{M}_{m,F,G} : \mathcal{H} \to \mathcal{H}$ is well defined and bounded with

$$\|\mathbf{M}_{m,F,G}\| \leq \|m\|_\infty \sqrt{B_F B_G}.$$

- Appropriate bilinear extension is given as follows.
- Let $\mathcal{H}$ be the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of complex Hilbert spaces, and $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with $\sigma-$finite positive measures μ_1, μ_2. Also, let $F = F_1 \otimes F_2$ and $G = G_1 \otimes G_2$ be continuous frames for $\mathcal{H}$ with respect to (X, μ) and $m : X \to \mathbb{C}$ be a measurable function. The operator $\mathbf{M}_{m,F,G} : \mathcal{H} \to \mathcal{H}$ weakly defined by

$$\begin{aligned} \langle \mathbf{M}_{m,F,G}\vec{f}, \vec{g}\rangle &= \langle \mathbf{M}_{m,F_1\otimes F_2,G_1\otimes G_2}\vec{f}, \vec{g}\rangle \\ &= \int_{X_1}\int_{X_2} m(x_1, x_2)\langle \vec{f}, (F_1 \otimes F_2)(x)\rangle\langle (G_1 \otimes G_2)(x), \vec{g}\rangle d\mu(x) \qquad (2) \end{aligned}$$

for all $\vec{f}, \vec{g} \in \mathcal{H}$, is called continuous bilinear frame multiplier of F and G with respect to the *symbol* m.

- If F (or G) is bounded, and if m is an essentially bounded measurable function with support of finite measure, then $\mathbf{M}_{m,F,G}$ is compact operator.
- Let $F = F_1 \otimes F_2$ and $G = G_1 \otimes G_2$ be norm bounded continuous frames for $\mathcal{H}$ with respect to (X, μ) and let $m \in L^1(X, \mu)$. Then $\mathbf{M}_{m,F,G}$ is a well-defined bounded bilinear operator. Moreover, $\mathbf{M}_{m,F,G}$ is a trace class operator.
- Finally, note that bilinear localization operators considered in

Teofanov, N., Bilinear Localization Operators on Modulation Spaces, Journal of Function Spaces (2018) doi: 10.1155/2018/7560870

can be interpreted as bilinear continuous frame multipliers.

- More details will be available in

Balazs P., Teofanov, N., Bilinear multipliers for continuous frames, *in preparation*

conclusion of part III

- Localization operators are phase-space analogues of localization techniques related to Fourier multipliers. They appear in different context such as in quantization, optics, signal analysis.
- Continuous frame multipliers offer an abstract approach to the study of localization operators.

Thank you for your kind attention.