Multiresolution expansions and Wavelet series in Gelfand-Shilov spaces

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- part I
  Gelfand-Shilov spaces
- part II
  highly regular multiresolution analysis
- part III
  multiresolution expansions in Gelfand-Shilov and their dual spaces

part I

- In the preface to the classic book *Generalized Functions, Vol. 2*, I.M. Gel'fand and G.E. Shilov wrote:
  "The fundamental peculiarity of the theory of generalized functions is that *different classes of problems require different classes of spaces*, and, indeed classes of spaces and not individual spaces."[1]
- For example, in the study of travelling waves (and other related problems in applied mathematics) as essential feature of solutions is their exponential decay and holomorphic extension property.
  Gelfand-Shilov type spaces provide a precise language to describe such phenomena.
  A neat exposition of the subject can be found in

F. Nicola, L. Rodino, Global Pseudo-differential calculus on Euclidean spaces, Pseudo-Differential Operators. Theory and Applications 4, Birkhäuser Verlag, Basel, 2010.

[1] I.M. Gel'fand, G.E. Shilov, *Generalized Functions, Vol. 2*, AMS 1968

- For linear partial differential/pseudodifferential operators P with analytic coefficients, and $Pu = f$, where f is smooth and of a rapid decay, the main goal is to find optimal conditions on P guaranteeing that u satisfies certain uniform regularity and decay estimates.
- Another issue is to find optimal conditions on P and the nonlinear term $F(u)$ so that the same holds for the solution u of the equation

$$P(x,D)u(x) = f(x) + F(u), \quad x \in \mathbb{R}^d.$$

The particular case $Pu = F(u)$ is relevant for the study of qualitative properties of solitary wave-type solutions of semilinear equations in mathematical physics.

- We refer to

T. Gramchev, Gelfand–Shilov Spaces: Structural Properties and Applications to Pseudodifferential Operators in $\mathbb{R}$, 1–68, in Quantization, PDEs, and Geometry, The Interplay of Analysis and Mathematical Physics, Birkhäuser, 2016.

for a recent survey of the subject.

- Let $\rho_1, \rho_2 \geq 0$ and $f \in C^\infty(\mathbb{R}^d)$. Then $f \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ if $(\exists C > 0)$ $(\exists h > 0)$ $(\exists \varepsilon > 0)$

$$\left|\partial^\beta f(x)\right| \leq C h^{|\beta|} \beta!^{\rho_1} e^{-\varepsilon |x|^{1/\rho_2}}, \quad \forall x \in \mathbb{R}^d, \quad \forall \beta \in \mathbb{N}_0^d. \tag{1}$$

- It is a worthwhile exercise to prove that $f \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ if and only if $(\exists C > 0)$ $(\exists h > 0)$

$$\sup_{x \in \mathbb{R}^d} \left|x^\alpha \partial^\beta f(x)\right| \leq C h^{|\alpha|+|\beta|} \alpha!^{\rho_2} \beta!^{\rho_1}, \quad \forall \alpha, \beta \in \mathbb{N}_0^d \tag{2}$$

- If (1) (or (2)) holds *for every* $h > 0$ then $f \in \Sigma_{\rho_2}^{\rho_1}(\mathbb{R}^d)$.[2]

[2] This (projective limit w.r.t. to h) is referred to as the Beurling case.

- $\mathcal{S}_{\rho_2}^{\rho_1}$ and $\Sigma_{\rho_2}^{\rho_1}$ are not trivial *iff* $\rho_1 + \rho_2 > 1$, or $\rho_1 + \rho_2 = 1$ and $\rho_1 \rho_2 > 0$, except $\Sigma_{1/2}^{1/2} = \emptyset$.
- Let $\rho_1 + \rho_2 \geq 1$. Then $f \in \mathcal{S}_{\rho_2}^{\rho_1}$ extends to an entire function if $\rho_1 < 1$, and to a holomorphic function in a strip if $\rho_1 = 1$.
- If $\rho_1 + \rho_2 < 1$, then $\mathcal{S}_{\rho_2}^{\rho_1} = \{0\}$.
- Whenever nontrivial, such spaces contain "enough functions":
  A test function space Φ is "rich enough" if

$$\int f(x)\varphi(x)dx = 0, \quad \forall \varphi \in \Phi \Rightarrow f(x) \equiv 0 (a.e.).$$

- For example, Hermite functions

$$H_n(t) = (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} \exp(t^2/2)(\exp(-t^2))^{(n)}, \quad t \in \mathbb{R}, \quad n = \mathbb{N},$$

belong to any nontrivial Gelfand–Shilov Space.

The following conditions are equivalent:

- $f \in \mathcal{S}^{\rho_1}_{\rho_2}$ (resp. $f \in \Sigma^{\rho_1}_{\rho_2}$, $\rho_1 > 1/2$) i.e., $(\exists C > 0)$ $(\exists h > 0)$

$$\sup_{x \in \mathbb{R}^d} \left| x^{\alpha} \partial^{\beta} f(x) \right| \leq C h^{|\alpha|+|\beta|} \alpha!^{\rho_2} \beta!^{\rho_1}, \quad \forall \alpha, \beta \in \mathbb{N}_0^d;$$

- $\sup_{x \in \mathbb{R}^d} |x^{\alpha} f(x)| \leq C h^{\alpha} \alpha!^{\rho_2}$ and $\sup_{x \in \mathbb{R}^d} |\partial^{\beta} f(x)| \leq C k^{\beta} \beta!^{\rho_1}$;
- $\sup_{x \in \mathbb{R}^d} |x^{\alpha} f(x)| \leq C h^{\alpha} \alpha!^{\rho_2}$ and $\sup_{\xi \in \mathbb{R}^d} |\xi^{\beta} \hat{f}(\xi)| \leq C k^{\beta} \beta!^{\rho_1}$;
- $\sup_{x \in \mathbb{R}^d} |f(x)| e^{h|x|^{1/\rho_2}} < \infty$ and $\sup_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)| e^{k|\xi|^{1/\rho_1}} < \infty$;
  for some $C > 0$, and for some (resp. for every) $h, k > 0$.

Here, $\hat{f}$ denotes the Fourier transform of f: $\hat{f}(\xi) = \int f(t) e^{-2\pi i t \xi} dt$.
It follows that the triviality of $\mathcal{S}^{\rho_1}_{\rho_2}$ for $\rho_1 + \rho_2 < 1$ can be considered as a version of the Heisenberg uncertainty principle.

- This characterization tells us that the Gelfand-Shilov type spaces might be a useful tool in time-frequency analysis.
- For given $\rho_1, \rho_2 \geq 0$, the Fourier transform is a topological isomorphism between $S^{\rho_1}_{\rho_2}$ and $S^{\rho_2}_{\rho_1}$ ($\mathcal{F}(S^{\rho_1}_{\rho_2}) = S^{\rho_2}_{\rho_1}$) which can be extended to a continuous linear transform from $(S^{\rho_2}_{\rho_1})'$ onto $(S^{\rho_2}_{\rho_1})'$.
- Compactly supported Gevrey functions $\mathcal{S}^{\rho_1}_0 = \mathcal{D}^{\rho_1}$ form the subspace of $\mathcal{S}^{\rho_1}_{\rho_2}$ ($\rho_1 > 1$).
- For the study of the range of the wavelet transform we introduce Gelfand-Shilov spaces in the upper half-plane $\mathbb{H}^{d+1} = \mathbb{R}^d \times \mathbb{R}_+$ as follows:
  Let $s, t, \tau_1, \tau_2 > 0$. A smooth function Φ belongs to $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{d+1})$ if for every $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ there exist constants $h, \varepsilon > 0$ such that

$$\left|\partial_a^\alpha \partial_b^\beta \Phi(b,a)\right| \leq h^{|\beta|+1} \beta!^s \, e^{-\varepsilon\left(a^{1/\tau_1} + a^{-1/\tau_2} + |b|^{1/t}\right)} .$$

- We consider the closed subspace of $\mathcal{S}^{\rho_1}_{\rho_2}$ consisting of functions with all vanishing moments:

$$(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^\alpha \varphi(x) \mathrm{d}x = 0, \ \forall \alpha \in \mathbb{N}^d \right\}. \quad (3)$$

- $(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^d)$ is non-trivial if and only if $\rho_2 > 1$.
  Indeed, $\hat{\varphi} \in \mathcal{S}^{\rho_2}_{\rho_1}(\mathbb{R}^d)$, and on the Fourier transform side (3) becomes $\hat{\varphi}^{(\alpha)}(0) = 0$, $\alpha \in \mathbb{N}^d$. Then $\hat{\varphi} \equiv 0$ if $\rho_2 \leq 1$, that is $\varphi \equiv 0$ if $\rho_2 \leq 1$, and the assertion follows.

conclusion of part I

- Gelfand-Shilov spaces are defined through different types of exponential decay and global highly regular features of its elements.
- The broad family of Gelfand-Shilov spaces can replace of the Schwartz space of rapidly decreasing functions $\mathcal{S}$ when more refinement properties of signals in time-frequency plane should be detected.
- Their dual spaces of (ultra)distributions can be used in the analysis of exponential growth instead. This is out of reach of the dual of $\mathcal{S}$, i.e. the space of tempered distributions.

part II

- In this lecture we celebrate 30 years (*pearl jubilee*) since the concept of multiresolution approximation became public.

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MULTIRESOLUTION APPROXIMATIONS AND WAVELET ORTHONORMAL BASES OF $\mathbf{L}^2(\mathbf{R})$

STEPHANE G. MALLAT

ABSTRACT. A multiresolution approximation is a sequence of embedded vector spaces $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ for approximating $\mathbf{L}^2(\mathbf{R})$ functions. We study the properties of a multiresolution approximation and prove that it is characterized by a 2π-periodic function which is further described. From any multiresolution approximation, we can derive a function $\psi(x)$ called a wavelet such that $(\sqrt{2^j}\psi(2^j x - k))_{(k,j)\in\mathbf{Z}^2}$ is an orthonormal basis of $\mathbf{L}^2(\mathbf{R})$. This provides a new approach for understanding and computing wavelet orthonormal bases. Finally, we characterize the asymptotic decay rate of multiresolution approximation errors for functions in a Sobolev space $\mathbf{H}^s$.

- A multiresolution approximation (MRA) of $L^2(\mathbb{R}^d)$ is an increasing sequence $\{V_m\}_{m\in\mathbb{Z}}$ of its closed subspaces such that
  (i) $\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$ and $\bigcup_{m\in\mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R}^d)$;
  (ii) $f(x) \in V_m \Leftrightarrow f(2x) \in V_{m+1},\ m \in \mathbb{Z}$;
  (iii) $f(x) \in V_0 \Leftrightarrow f(x-n) \in V_0,\ n \in \mathbb{Z}^d$;
  (iv) there exists a *scaling function* $\phi \in L^2(\mathbb{R}^d)$ such that $\{\phi(x-n)\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of V_0.
- Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An MRA is called (ρ_1, ρ_2)-regular if it possesses a scaling function $\phi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$.
- If $\phi \in \mathcal{S}(\mathbb{R}^d)$, then $\hat{\phi}(0) = 1$ and $\partial^\alpha \hat{\phi}(0) = 0$ for any nonzero multi-index α.
- Thus the condition $\rho_2 > 1$ is dictated by the very nature of an MRA. Indeed, if $\rho_2 < 1$, then the scaling function of a (ρ_1, ρ_2)-regular MRA would be identically equal to 1 due to the analyticity of the elements of $\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ when $\rho_2 < 1$.

- Recall, a function $\psi \in L^2(\mathbb{R})$ is called an *orthonormal wavelet* if the set $\{\psi_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{m,n}(x) = 2^{\frac{m}{2}} \psi(2^m x - n), \qquad m, n \in \mathbb{Z},\ x \in \mathbb{R}.$$

- A powerful way to construct orthonormal wavelets is via MRA. Note that any orthonormal wavelet belonging to $\mathcal{S}(\mathbb{R})$ arise from an MRA. Moreover, all of its moments must vanish.
- Next we give an example of a scaling functions which defines a (ρ_1, ρ_2)-regular MRA, together with the corresponding wavelet.
- The construction is essentially due to Lemarié-Meyer,

P. G. Lemarié, Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana **2** (1986), 1–18.

- Let $\rho_2 > 1$, and $a < \pi/3$. Then we choose $\varphi \in \mathcal{D}^{\rho_2}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subseteq [-a, a]$ and $\int_{-\infty}^{\infty} \varphi(\xi)\, d\xi = \pi/2$. Put $\varphi_2(\xi) = (1/2)\varphi(\xi/2)$.
- Consider the bell type function

$$b(\xi) = \sin\left(\int_{-\infty}^{\xi-\pi} \varphi(t)dt\right) \cos\left(\int_{-\infty}^{\xi-2\pi} \varphi_2(t)dt\right), \quad \xi > 0,$$

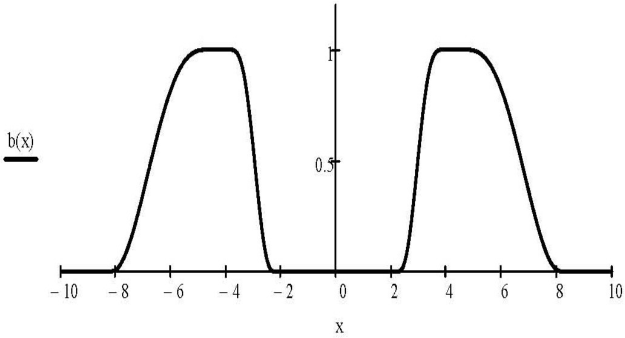
and extend it evenly to $(-\infty, 0]$. Then, $b \in \mathcal{D}^{\rho_2}(\mathbb{R})$ and

$$\operatorname{supp} b \subseteq [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3].$$

- The associated orthonormal wavelet is then given by

$$\hat{\psi}(\xi) = e^{i\xi/2} b(\xi), \quad \xi \in \mathbb{R}.$$

- We have $\psi \in \mathcal{F}(\mathcal{D}^{\rho_2}(\mathbb{R})) \subseteq \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R})$ for all $\rho_1 \geq 0$. Moreover, since $0 \notin \operatorname{supp} \hat{\psi}$, we obtain $\psi \in (\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R})$.

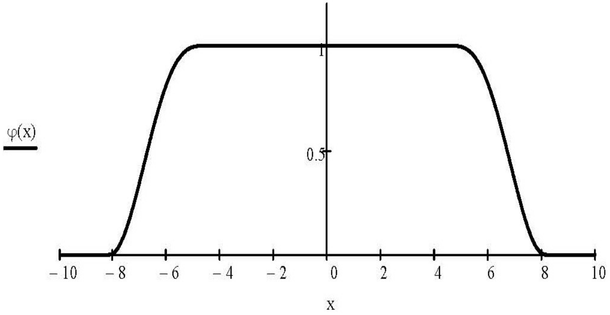

b(x)
1
0.5
– 10
– 8
– 6
– 4
– 2
0
2
4
6
8
10
x

- To select an associated scaling function we can choose

$$|\hat{\phi}(\xi)|^2 = \begin{cases} 1 & \text{if} \quad |\xi| \le 2\pi/3, \\ b^2(2\xi) & \text{if} \quad 2\pi/3 \le |\xi| \le 4\pi/3, \\ 0 & \text{if} \quad |\xi| \ge 4\pi/3, \quad \xi \in \mathbb{R}, \end{cases}$$

  and take $\arg \hat{\phi}(\xi) = \xi$, that is, the smooth function $\hat{\phi}(\xi) = e^{i\xi}|\hat{\phi}(\xi)|$.[3]
- Since $b \in \mathcal{D}^{\rho_2}(\mathbb{R})$ and $\hat{\phi} \in \mathcal{D}(\mathbb{R})$, we obtain $\hat{\phi} \in \mathcal{D}^{\rho_2}(\mathbb{R})$ and conclude that $\phi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R})$ for any $\rho_1 \ge 0$.
- This construction gives a (ρ_1, ρ_2)-regular MRA of $L^2(\mathbb{R})$.
- Tensoring the associated one dimensional scaling functions, one obtains examples of (ρ_1, ρ_2)-regular MRA of $L^2(\mathbb{R}^d)$.

[3] See also E. Hernández, G. Weiss, *A first course on wavelets*, CRC Press, 1996.


φ(x)
1
0.5
− 10
− 8
− 6
− 4
− 2
0
2
4
6
8
10
x

conclusion of part II

- It is well known that smooth orthonormal wavelets, with all derivatives bounded, cannot have exponential decay. We measure regularity and (subexponential) decay of orthonormal wavelets within the scale of Gelfand-Shilov spaces. Such wavelets necessarily have all vanishing moments.
- We essentially use the Lemarié-Meyer construction to obtain a highly regular scaling function.
- This framework is appropriate for the study of the effectiveness of MRA in the context of Gelfand-Shilov spaces.

part III

- Given an MRA $\{V_m\}_{m\in\mathbb{Z}}$, the orthogonal projection operator $q_0 : L^2(\mathbb{R}^d) \to V_0$ is determined by its kernel

$$q_0(x,y) = \sum_{k\in\mathbb{Z}^d} \phi(x-k)\bar{\phi}(y-k), \qquad x,y \in \mathbb{R}^d,$$

i.e.,

$$(q_0 f)(x) = \langle f(y), q_0(x,y)\rangle = \int_{\mathbb{R}^d} f(y) q_0(x,y)\, dy, \qquad x \in \mathbb{R}^d.$$

- Then, the orthogonal projection $q_m : L^2(\mathbb{R}^d) \to V_m$ is given by

$$(q_m f)(x) = \langle f(y), q_m(x,y)\rangle = \int_{\mathbb{R}^d} f(y) q_m(x,y)\, dy, \qquad x \in \mathbb{R}^d,$$

where $q_m(x,y) = 2^{md} q_0(2^m x, 2^m y)$.

- The sequence $\{q_m f\}_{m\in\mathbb{Z}}$ is called the multiresolution expansion of f.

- If a MRA is (ρ_1, ρ_2)-regular, then there exist constants $c > 0$ and $h > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta q_0(x,y)| \lesssim h^{|\alpha+\beta|} \alpha!^{\rho_1} \beta!^{\rho_1} \, e^{-c|x-y|^{\frac{1}{\rho_2}}}, \qquad \alpha, \beta \in \mathbb{N}^d, \; x, y \in \mathbb{R}^d.$$

- Therefore, for each fixed x, we have $q_m(x, \,\cdot\,) \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$. This allows one to define each $q_m f$ even for ultradistributions $f \in (\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d))'$ via the dual pairing $(q_m f)(x) = \langle f(y), q_m(x,y) \rangle$.

Theorem

Let $\rho_2 > 1$, $\rho_1 \geq 0$, and let $\{V_m\}_{m \in \mathbb{Z}}$ be a (ρ_1, ρ_2)-regular MRA. Set $\sigma = \rho_1 + \rho_2 - 1$ and let $s \geq \sigma$ and $t \geq \rho_2$.

(i) If $\varphi \in \mathcal{S}_t^{s-\sigma}(\mathbb{R}^d)$, then $\lim_{m\to\infty} q_m \varphi = \varphi$ in $\mathcal{S}_t^s(\mathbb{R}^d)$.

(ii) If $f \in ((\mathcal{S}_t^s)(\mathbb{R}^d))'$, then $\lim_{m\to\infty} q_m f = f$ in $(\mathcal{S}_t^{s-\sigma}(\mathbb{R}^d))'$.

- In the proof we reconsider Meyer's powerful method used in the proof of effectiveness of MRA in the case of limited regularity, cf.

Y. Meyer, *Wavelets and operators*, Cambridge University Press, Cambridge, 1992.

- The proof is based on careful comparison of each q_j with the convolution operator with kernel $\eta_m(x) = 2^{md}\eta(2^m x)$. Here, $\eta \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^d)$ is an even function such that $\int_{\mathbb{R}^d} \eta(x)\, dx = 1$.
- Due to the different nature of the involved regularity and decay properties the proof contains nontrivial modifications of Meyer's technique and arguments.
- Notice that we detect a loss of regularity, which can be quantitatively measured by an additional parameter.
- The same phenomena appears when considering the convergence of the wavelet series.

- For the construction of wavelet bases of $L^2(\mathbb{R}^d)$, we follow the tensor product approach, which we briefly recall.
- Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R})$ be a (ρ_1, ρ_2)-regular orthonormal wavelet, with the scaling function $\phi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R})$.
- Put $\psi_0(u) = \phi(u)$ and $\psi_1(u) = \psi(u)$, $u \in \mathbb{R}$, and let

$$\psi_\epsilon(x) = \psi_{\epsilon_1}(x_1)\psi_{\epsilon_2}(x_2)\cdots\psi_{\epsilon_d}(x_d), \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d,$$

where $\epsilon \in Q = \{0,1\}^d \setminus (0,\ldots,0)$. This gives $2^d - 1$ functions ψ_ϵ. Finally, for each $\lambda = (\epsilon, m, n) \in \Lambda = Q \times \mathbb{Z} \times \mathbb{Z}^d$, we set

$$\psi_\lambda(x) = \psi_{\epsilon,m,n}(x) \quad = \quad 2^{md/2}\psi_\epsilon(2^m x - n), \qquad x \in \mathbb{R}^d.$$

- Then, the collection of functions

$$\{\psi_\lambda \mid \lambda \in \Lambda\} = \{\psi_{\epsilon,m,n} \mid \epsilon \in Q, m \in \mathbb{Z}, n \in \mathbb{Z}^d\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

- Every function $f \in L^2(\mathbb{R}^d)$ can than be expanded as

$$f = \sum_{\lambda\in\Lambda} c^{\psi}_{\lambda}(f)\psi_{\lambda} = \sum_{\lambda\in\Lambda}\langle f, \bar{\psi}_{\lambda}\rangle\psi_{\lambda}. \tag{4}$$

The wavelet coefficients $c^{\psi}_{\lambda}(f)$ are well-defined for any $f \in (\mathcal{S}^{\rho_1}_{\rho_2})'_0(\mathbb{R}^d)$ by duality.

Theorem

Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R})$ be a (ρ_1, ρ_2)-regular orthonormal wavelet, where $\rho_1 \geq 0$ and $\rho_2 > 1$. Set $\sigma = \rho_1 + \rho_2 - 1$ and consider parameters $s > \sigma$ and $t > \sigma + 1$.

(i) If $\varphi \in (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}^d)$, then $\varphi = \sum_{\lambda\in\Lambda} c^{\psi}_{\lambda}(\varphi)\,\psi_{\lambda}$ converges in $(\mathcal{S}^{s}_{t})_0(\mathbb{R}^d)$.

(ii) If $f \in ((\mathcal{S}^{s}_{t})_0(\mathbb{R}^d))'$ then its wavelet series expansion (4) converges in (the strong dual topology of) $((\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}^d))'$.

(iii) The Parseval identity $\langle f, \varphi\rangle = \sum_{\lambda\in\Lambda} c^{\psi}_{\lambda}(f)\, c^{\bar{\psi}}_{\lambda}(\varphi)$ holds for any $f \in ((\mathcal{S}^{s}_{t})_0(\mathbb{R}^d))'$ and $\varphi \in (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}^d)$.

conclusion of part III: to take home

- Y. Meyer observed convergence properties of multiresolution expansions when the scaling function has certain regularity combined with polynomial decay.
- It is not surprising that his method could be used to show appropriate result in the presence of more regular scaling function with faster decay.
- We first set the stage by considering Gelfand-Shilov spaces of functions with exponential type decay in time-frequency plane, and MRA given by highly regular scaling functions. Then we perform nontrivial modifications of Meyer's method to obtain desired asymptotic properties of multiresolution expansions.
- Furthermore, we use orthonormal wavelets arising from a (ρ_1, ρ_2)-regular MRA and prove the convergence of related wavelet series expansions.

for your kind attention!