Asymptotic analysis and distributions
- some historical remarks -

Dedicated to the memory of Arpad Takači

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Excerpts from the biography of Professor Arpad Takači:

1979 Study stay in Kiel, Germany

1982 PhD thesis, under the supervision of Prof. Bogoljub Stanković

1985 Study stays in London and Leicester, England

1986 Study stays in Bordeaux and Paris, France

1985 (until 1989) Director of the Institute of Mathematics in Novi Sad

1990 the ”blue book”, *Asymptotic Behaviour and Stieltjes Transformation of Distributions*

1991 (until 1993) Director of the Institute of Mathematics in Novi Sad

1992 Fulbright Scholar at the Virginia Commonwealth University, Richmond, USA

- Scientific interests of Prof. Takači in the 1980s are reflected in a dozen papers published in that period and summarized in the book

B. Stanković, S. Pilipović, A. Takači, Asymptotic Behaviour and Stieltjes Transformation of Distributions, Teubner-Texte zur Mathematik 11, Leipzig, 1990.

- The book is a survey on many definitions of the asymptotic behaviour of distributions, their comparison, new results, and application to different problems.
- ”This book is a remarkable synthesis of the results of the last two decades concerning the asymptotic behaviour of distributions and their application to the Stieltjes and Fourier transformation, to partial differential equations and to the Abelian and Tauberian type theorems.” (Math Rev)
- Since the beginning of the XXI century it became (and remained) his most cited reference.

- The story about the main subject of the lecture may begin with Littlewood's generalization of Tauber's theorem[1] published in 1911,

  J. E. Littlewood, The converse of Abels theorem on power series, Proc. London Math. Soc. 2 (1911), 434–448

- The importance of the Littlewood paper is well explained in e.g.

  D. Choimet, H. Queffélec, Twelve landmarks of twentieth-century analysis, Cambridge University Press, 2015.

  J. Korevaar, Tauberian Theory, a Century of Developments, Berlin, Springer, 2004.

  N. Wiener, Tauberian theorems, Annals of Math. 33 (1932) 1–100

- Recall, an Abelian theorem is any statement to the effect that a method of summability assigns to each convergent series its ordinary sum.
- A Tauberian theorem is the converse of an Abelian theorem under a supplementary growth condition.
- In other words: "summability" by a certain method + Tauberian condition ⇒ "summability" by other methods.

[1] "The first deep Tauberian theorem", J. J. Benedetto in his Spectral synthesis, 1975

- As the prototype for a class of theorems now known as Abelian theorems[2] we recall the original result from 1826 about the convergence of power series:

$$\sum_{n=0}^{\infty} a_n = s \quad \Longrightarrow \quad \lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = s.$$

- The converse of Abel's theorem is false (in general). Indeed

$$1 - x + x^2 - x^3 + \dots \quad = \frac{1}{1+x} \to \frac{1}{2}, \quad \text{as} \quad x \to 1^-$$

while the power series with $x = 1$ becomes a divergent series

$$1 - 1 + 1 - 1 + \dots$$

- There is an integral analogue of Abel's theorem that involves the Laplace transform: If g is a bounded and integrable function on $[0,\infty)$, then

$$\lim_{x\to 0^+} \int_0^\infty e^{-xt} g(t)dt = \int_0^\infty g(t)dt.$$

[2] P. Duren, Invitation to Classical Analysis, AMS, Providence, Rhode Island, 2012

- Although the series $\sum_{n=0}^{\infty}(-1)^n$ is divergent we may say that it is *Abel summable* to the sum $1/2$.
- A summability method should be regular and linear.[3]

1897 A. Tauber proved that under the condition $na_n \to 0$ as $n \to \infty$ we have

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = s \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n = s.$$

1911 Littlewood proved Hardy's conjecture that boundedness of $(na_n)_{n\in\mathbb{N}}$ is a sufficient condition for Tauber's claim.
This marks the beginning of their famous cooperation.

1932 Wiener's approach is based on the Fourier transform methods. It was a step forward leading to his famous *generalized harmonic analysis*.

- Wiener's theory can be used to prove the prime number theorem, and influenced Gelfand's work on Banach algebras, the fundament for Abstract Harmonic Analysis.

[3] A. I. Saichev, W. Woyczynski, Distributions in the Physical and Engineering Sciences, Volume 1, Birkhauser, Basel, 2018

- Wiener Tauberian theorem: Let $g \in L^1(\mathbb{R})$ have a nonvanishing Fourier transform and let $\varphi \in L^\infty(\mathbb{R})$. If

$$\lim_{x\to\infty}(g*\varphi)(x) = \lim_{x\to\infty}\int g(y)\varphi(x-y)dy = r\int g(y)dy$$

  then

$$\forall f \in L^1(\mathbb{R}), \quad \lim_{x\to\infty}(f*\varphi)(x) = r\int f(y)dy.$$

- Here, the Tauberian condition is $\varphi \in L^\infty(\mathbb{R})$, and g represents the summability method.
  The conclusion is summability for the whole class of summability methods, namely for all $f \in L^1(\mathbb{R})$.
- ”Wiener’s Tauberian theorem provides the proper notion of spectrum for phenomena such as white light...”[4]

[4] J.J. Benedetto, ANHA Series Preface

1930 Karamata gave a surprisingly simple proof of Hardy-Littlewood's theorem which brought him international recognition.

- "After Hardy, Littlewood and Wiener, the principal actor in Tauberian theory was Karamata... his heritage, involving *regular variation* became indispensable in asymptotics of all kinds... Regular variation is now a subject in its own right...
  Regularly varying functions play a role in many parts of mathematics: number theory, complex analysis, Tauberian theory, differential equations, and especially probability theory."[5]
- A function L defined on some interval (a, ∞) or $[a, \infty)$, $a \geq 0$, is called *slowly varying at infinity* if it is measurable, positive and

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1, \quad \text{for every number } \ a > 0.$$

[5] J. Korevaar in Tauberian Theory, 2004

- Functions $\phi(x)$ of the form $x^\alpha L(x)$ with real α are *regularly varying at infinity* (of index α). Such functions are characterized by the relation

$$\lim_{x\to\infty} \frac{\phi(ax)}{\phi(x)} = a^\alpha, \quad \text{for every number } \ a > 0.$$

- Slowly (and regularly) varying functions at the origin (or at $x_0 \in \mathbb{R}$) are defined analogously.
- The slow and regular variation are preserved under summation and integration. It turns out that for a large class of functions f,

$$\int_0^\infty f(x)L(ax)dx = L(a) \int_0^\infty f(x)dx, \quad a > 0.$$

- Karamata proved Abelian and Tauberian theorems involving regular variation for Laplace and Stieltjes transform.
- Laplace transform is related to fractional derivation and integration via the powers of the Cauchy-Szegö kernel, and the Stieltjes transform can be viewed as iterated Laplace transform (see the ”blue book”).

1954 PhD thesis of Bogoljub Stanković (advisor J. Karamata).

1979 PhD thesis of Stevan Pilipović (advisor, B. Stanković).

1982 PhD thesis of Arpad Takači (advisor, B. Stanković).

- It turned out that slowly varying functions are the natural choice for comparison functions in *the asymptotic analysis of generalized functions*. This fact is justified in the work of Stanković, Pilipović and Takači in the 1980s.
- The study of asymptotic behavior of distributions traces back to the work of Lighthill (1958), Silva (1964), Lavoine and Misra (1974).
- An important step forward was the notion of quasi-asymptotic behavior introduced by Zavyalov (1973).[6] The (early) work of Vladimirov, Drozhzhinov and Zavyalov was motivated by questions in quantum field theory, and summarized in

V. S. Vladimirov, Yu. N. Drozzinov, B.I. Zavialov, Tauberian theorems for generalized functions, Kluwer Academic Publishers Group, Dordrecht, 1988.

- Another approach to the asymptotic behavior of distributions is related to the so called moment expansions in the work of Estrada, Kanwal, Vindas...

[6] Zavyalov used the term ”automodel asymptotics”

- *Quasi-asymptotic behaviour* at infinity can be characterized by a Tauberian theorem for the Laplace transform of a (tempered) distribution.
- The space of tempered distributions is a "natural" one for the quasi-asymptotics.
- *S-asymptotic behaviour* at infinity, systematically studied in the "blue book" is convenient for the space of exponential distributions $\mathcal{K}_1'$.
- The problem of comparison of different types of asymptotic behaviours (and their "natural" spaces of distributions) is treated in the "white" book

S. Pilipović, B. Stanković, J. Vindas, Asymptotic behavior of generalized functions, World Scientific, Hackensack, NJ, 2012.

- Abelian and Tauberian theorems for quasiasymptotics and multiresolution expansions are discussed in

  S. Pilipović, A. Takači, N. Teofanov, Wavelets and quasiasymptotics at a point. J. Approx. Theory 97 (1999), 40–52

- A complete study of quasi-asymptotics of distributions through the wavelet transform is performed in

  J. Vindas, S. Pilipović, D. Rakić, Tauberian theorems for the wavelet transform. J. Fourier Anal. Appl. 17 (2011), 65–95.

- This is done via Abelian-Tauberian type results which imply intrinsic properties of functions as well.
- In particular, the wavelet transform characterization of quasi-asymptotic behaviour of tempered distributions is used to prove the Littlewood's Tauberian theorem.
- This enlightens the use of theory of distributions in investigations of local properties of functions.
- For another interesting issue, namely the quantified Tauberian theorems, we refer to recent work of J. Vindas and his group.