Boundary values and wave front sets

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Dedicated to the memory of Professor Arpad Takači

It is well known that (ultra)distributions can be represented as boundary values of analytic functions. The essence of the problem is to determine under which conditions the analytic function $F(x+iy)$ (observed on a certian conical domain $\Gamma \subset \mathbf{R}^d$ with respect to y) is a well defined (ultra)distribution as y tends to 0 in Γ.

The classical result of Hörmander can be roughly interpreted as follows: If

$$F(x+iy) \leq C|y|^{-M}, \quad y \in \Gamma,$$

for some $C, M > 0$ then $F(x+i\Gamma 0)$ is in the Schwartz space $\mathcal{D}'(U)$ in a neighborhood $U \subset \mathbf{R}^d$ of x.

By $M_p = (M_p)_{p\in\mathbf{N}}$, $M_0 = 1$, we denote a sequence of positive numbers such that the following conditions hold:

$$(M.1) \qquad M_p^2 \leq M_{p-1}M_{p+1}, \qquad p \in \mathbf{N};$$

$$(M.2) \quad (\exists C > 0) \;\; M_{p+q} \leq C^{p+1} M_p M_q, \quad p, q \in \mathbf{N};$$

$$(M.3)' \qquad \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

Let $M_p^* = \dfrac{M_p}{p!}$. Then M_p satisfies condition $(M.1)^*$ if M_p^* satisfies $(M.1)$, i.e,

$$(M.1)^* \quad \left(\frac{M_p}{p!}\right)^2 \leq \frac{M_{p-1}}{(p-1)!}\frac{M_{p+1}}{(p+1)!}, \quad p \in \mathbf{N},$$

$$(M.1)^* \Rightarrow (M.1).$$

Example

$M_p = p!^t$, $t > 1$, satisfies conditions $(M.1)^* - (M.3)'$.

Let $U \subseteq \mathbf{R}^d$ be open, $K \subset\subset U$ and $h > 0$. $\phi \in C^\infty(U)$ belongs to a Banach space $\mathcal{E}_{M_p,h}(K)$ if

$$\|\phi\|_{\mathcal{E}_{M_p,h}(K)} = \sup_{\alpha \in \mathbf{N}^d, x \in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty.$$

By $\mathcal{D}^K_{M_p,h}$ we denote its subspace of functions supported in K.
Locally convex spaces of *ultradifferentiable functions of class M_p* are given by

$$\mathcal{E}_{M_p}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \to \infty} \mathcal{E}_{M_p,h}(K) = \bigcap_{K \subset\subset U} \bigcup_{h > 0} \mathcal{E}_{M_p,h}(K),$$

$$\mathcal{D}_{M_p}(U) = \varinjlim_{K \subset\subset U} \varinjlim_{h \to \infty} \mathcal{D}^K_{M_p,h} = \bigcup_{K \subset\subset U} \bigcup_{h > 0} \mathcal{D}^K_{M_p,h},$$

and the strong dual of $\mathcal{D}_{M_p}(U)$, denoted by $\mathcal{D}'_{M_p}(U)$, is the space *ultradistributions of class M_p*.

Associated function to the seuence M_p, $p \in \mathbf{N}$, is defined by

$$T_{M_p}(k) = T(k) = \sup_{p \in \mathbf{N}} \ln \frac{k^p}{M_p}, \quad k > 0.$$

Moreover,

$$T_{M_p^*}(k) = T^*(k) = \sup_{p \in \mathbf{N}} \ln \frac{p!k^p}{M_p}, \quad k > 0.$$

Example

If $M_p = p!^t$, $t > 1$, then there exists $A, B > 0$ such that

$$Ak^{1/t} - B \leq T(k) \leq Ak^{1/t}, \quad k > 0.$$

$$Ak^{1/(t-1)} - B \leq T^*(k) \leq Ak^{1/(t-1)}, \quad k > 0.$$

Theorem

(Komatsu, Pilipović etc.) Let M_p, $p \in \mathbf{N}$, satisfies conditions $(M.1)^* - (M.3)'$. Moreover, let U be open set in $\mathbf{R}^d$, Γ open cone in $\mathbf{R}^d$ and $\gamma > 0$. If $F(z)$ is an analytic function in

$$Z = \{z \in \mathbf{C}^d \,|\, \mathrm{Re}\, z \in U,\ \mathrm{Im}\, z \in \Gamma, |\mathrm{Im}\, z| < \gamma\},$$

such that for every $h > 0$ there exists $A_h > 0$

$$|F(z)| \leq A_h e^{T^*(h/|y|)} \quad z = x + iy \in Z,$$

then

$$F(x+iy) \to F(x+i0), \quad y \to 0,\ y \in \Gamma,\ \text{in } \mathcal{D}'_{M_p}(U).$$

Let $\tau > 0$, $\sigma > 1$, $h > 0$. A smooth function ϕ on U belongs to the Banach space $\mathcal{E}_{\tau,\sigma,h}(K)$ if

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}} < \infty \,,$$

Let $\mathcal{D}^K_{\tau,\sigma,h}$ be the set of functions in $\mathcal{E}_{\tau,\sigma,h}(K)$ with the supports in K. Then, we set

$$\mathcal{E}_{\tau,\sigma}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K) \quad \mathcal{D}_{\tau,\sigma}(U) = \varinjlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h} \,.$$

By $\mathcal{D}'_{\tau,\sigma}(U)$ we denote the dual of $\mathcal{D}_{\tau,\sigma}(U)$.

Proposition

$$\varinjlim_{t\to\infty} \mathcal{G}_t(U) \hookrightarrow \mathcal{E}_{\tau,\sigma}(U) \hookrightarrow C^\infty(U),$$

where $\mathcal{G}_t(U)$ is Gevrey class with index t.

Lemma

Let $\tau > 0$, $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}\backslash\{0\}$, $M_0^{\tau,\sigma} = 1$. Then $M_p^{\tau,\sigma}$ satisfies $(M.1)^*$ and $(M.3)'$. Moreover, for some $C > 0$ we have:

$\widetilde{(M.2)'}$ $M_{p+1}^{\tau,\sigma} \leq C^{p^\sigma+1} M_p^{\tau,\sigma}$, $p \in \mathbf{N}$,

$\widetilde{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma+q^\sigma+1} M_p^{\tau 2^{\sigma-1},\sigma} M_q^{\tau 2^{\sigma-1},\sigma}$, $p, q \in \mathbf{N}$.

Let $m_p^{\tau,\sigma} = \dfrac{M_p^{\tau,\sigma}}{M_{p-1}^{\tau,\sigma}}$, $p \in \mathbf{N}$. Then

$$M_p^{\tau,\sigma} \leq (m_p^{\tau,\sigma})^p \leq C^{p^\sigma} M_p^{\tau(2^\sigma-1),\sigma}, \quad p \in \mathbf{N},$$

where $C > 0$ is constant appearing in $\widetilde{(M.2)}$.

Definition

Let $\tau > 0$, $\sigma > 1$. A differential operator of infinite order $P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)\partial^\alpha$ is said to be an *ultradifferential operator of class* τ, σ on an open set $U \subset \mathbf{R}^d$, if for every $K \subset\subset U$ there exists $h > 0$ and for every $L > 0$ there exists $A > 0$ such that,

$$\sup_{x\in K} |\partial^\beta a_\alpha(x)| \leq A h^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad \alpha, \beta \in \mathbf{N}^d.$$

Theorem

If $P(x,\partial)$ ultradifferential operator of class τ, σ, then

$$P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U),$$

is continuous linear mapping.

Let $h > 0$, $\tau > 0$ and $\sigma > 1$. The *extended associated function* related to the sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, is defined by

$$T_{\tau,\sigma,h}(k) = \sup_{p \in \mathbf{N}} \ln \frac{h^{p^\sigma} k^p}{M_p^{\tau,\sigma}}, k > 0,$$

and

$$T^*_{\tau,\sigma,h}(k) = \sup_{p \in \mathbf{N}} \ln \frac{p! h^{p^\sigma} k^p}{M_p^{\tau,\sigma}}, \quad h, k > 0.$$

Lemma

For any $h > 0$ there exists $H > h$ such that

$$T_{\tau,\sigma,h}(k) \le T^*_{\tau,\sigma,h}(k) \le T_{\tau,\sigma,H}(k), \quad k > 0.$$

Theorem

Let $h > 0$. There exists constants $B_1, B_2, b_1, b_2 > 0$ such that

$$B_1\, k^{b_1 \left(\frac{\ln k}{\ln(\ln k)}\right)^{\frac{1}{\sigma-1}}} \leq e^{T_{\tau,\sigma,h}(k)} \leq B_2\, k^{b_2 \left(\frac{\ln k}{\ln(\ln k)}\right)^{\frac{1}{\sigma-1}}}, \quad k > e.$$

More precisely, let

$$h > 0,\ c_1 = \left(\frac{\sigma-1}{\tau\sigma}\right)^{\frac{1}{\sigma-1}},\ c_2 = h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \frac{\sigma-1}{\tau\sigma}.$$

Then, there exist constants $A_1, A_2 > 0$ such that

$$A_1\, k^{\frac{1}{2}\frac{\sigma-1}{\sigma} c_1 \left(\frac{\ln k}{\ln(c_2 \ln k)}\right)^{\frac{1}{\sigma-1}}} \leq e^{T_{\tau,\sigma,h}(k)} \leq A_2\, k^{c_1 \left(\frac{\ln k}{\ln(c_2 \ln k)}\right)^{\frac{1}{\sigma-1}}}, \quad k > e.$$

Theorem

Let U be open set in $\mathbf{R}^d$, Γ open cone in $\mathbf{R}^d$ and $\gamma > 0$. Let $F(z)$ be an analytic function in

$$Z = \{z \in \mathbf{C}^d \,|\, \operatorname{Re} z \in U\,, \operatorname{Im} z \in \Gamma, |\operatorname{Im} z| < \gamma\},$$

such that for every $h > 0$ there exists $A_h > 0$

$$|F(z)| \leq A_h e^{T_{(2^{\sigma}-1)\tau,\sigma,h}(1/|y|)}, \quad z = x + iy \in Z.$$

Then

$$F(x+iy) \to F(x+i0), \quad y \to 0,\, y \in \Gamma, \text{ in } \mathcal{D}'_{\tau/2^{\sigma-1}}(U).$$

Definition

Let $u \in \mathcal{D}'_{\tau,\sigma}(u)$, $\tau > 0$, $\sigma > 1$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\tau,\sigma}(u)$ if and only if there exists a conic neighborhood Γ of ξ_0, a compact neighborhood K of x_0 and $\phi \in \mathcal{D}_{\tau,\sigma}(U)$, $\phi = 1$, $\operatorname{supp}\phi \subseteq K$, on a neighborhood of x_0, such that for some $A, h > 0$

$$|\widehat{\phi u}(\xi)| \leq A e^{-T_{\tau,\sigma,h}(|\xi|)}, \quad N \in \mathbf{N}, \xi \in \Gamma.$$

Proposition

For $\sigma > 1$ and $0 < \tau_1 < \tau_2$ we have

$$\mathrm{WF}(u) \subset \mathrm{WF}_{\tau_2,\sigma}(u) \subset \mathrm{WF}_{\tau_1,\sigma}(u) \subset \bigcap_{t>1} \mathrm{WF}_t(u), \quad u \in \mathcal{D}'(U),$$

where $\mathrm{WF}_t(u)$ denotes Gevrey wave front set.

For $\sigma > 1$

$$\mathcal{D}^{(\sigma)}(U) = \varprojlim_{\tau\to 0} \mathcal{D}_{\tau,\sigma}(U), \quad \mathcal{D}^{\{\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{D}_{\tau,\sigma}(U)$$

and let $\mathcal{D}'^{(\sigma)}(U)$, $\mathcal{D}'^{\{\sigma\}}(U)$ be the corresponding dual spaces.
Denote

$$\mathrm{WF}^{\{\sigma\}}(u) = \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \quad u \in \mathcal{D}'^{\{\sigma\}}(U),$$

and

$$\mathrm{WF}^{(\sigma)}(u) = \bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \quad u \in \mathcal{D}'^{(\sigma)}(U).$$

Corollary

Let $u \in \mathcal{D}'^{\{\sigma\}}(U)$ (resp. $\mathcal{D}'^{(\sigma)}(U)$), $\sigma > 1$. Then $(x_0, \xi_0) \notin \mathrm{WF}^{\{\sigma\}}(u)$ (resp. $(x_0, \xi_0) \notin \mathrm{WF}^{(\sigma)}(u)$) if and only if there exists a conic neighborhood Γ of ξ_0, a compact neighborhood K of x_0, and $\phi \in \mathcal{D}^{\{\sigma\}}(U)$ (resp $\phi \in \mathcal{D}^{(\sigma)}(U)$) such that $\operatorname{supp} \phi \subseteq K$, $\phi = 1$ on some neighborhood of x_0, and

$$|\widehat{\phi u}(\xi)| \leq A|\xi|^{-H\left(\frac{\ln|\xi|}{\ln(\ln|\xi|)}\right)^{\frac{1}{\sigma-1}}}, \quad \xi \in \Gamma,$$

for some $A, H > 0$ (resp. for any $H > 0$ there exists $A > 0$).

Theorem

Let $\sigma > 1$, U be open set in $\mathbf{R}^d$, Γ open cone in $\mathbf{R}^d$ and $\gamma > 0$. If $F(z)$ is an analytic function in

$$Z = \{z \in \mathbf{C}^d \,|\, \mathrm{Re}\, z \in U\,, \mathrm{Im}\, z \in \Gamma, |\mathrm{Im}\, z| < \gamma\},$$

such that

$$|F(z)| \leq A|y|^{-H\left(\frac{\ln(1/|y|)}{\ln(\ln(1/|y|))}\right)^{\frac{1}{\sigma-1}}}, \quad z = x + iy \in Z,$$

for some $A, H > 0$ (resp. for every $H > 0$ there exists $A > 0$). Then $u(x) = F(x + i\Gamma 0) \in \mathcal{D}'^{(\sigma)}(U)$ (resp. $\mathcal{D}'^{\{\sigma\}}(U)$).
Moreover,

$$\mathrm{WF}^{(\sigma)}(u) \subseteq U \times \Gamma_0, \quad (\textit{resp.}\ \mathrm{WF}^{\{\sigma\}}(u) \subseteq U \times \Gamma_0),$$

where $\Gamma^0 = \{\xi \in \mathbf{R}^d \,|\, y \cdot \xi \geq 0 \text{ for all } y \in \Gamma\}$.

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THANK YOU FOR YOUR ATTENTION!