# Dynamical Sampling on Graphs

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***Abstract*— Recently, the framework of Dynamical Sampling has been applied to combinatorial graphs. Setting our iteration matrix as the graph Laplacian, $M$, we have found conditions for stable reconstruction of Paley-Wiener spaces (denoted $PW_{\lambda_k}$ where $\lambda_k$ is some eigenvalue of $M$) of graphs via dynamical samples. Also, given $\tilde{y}_i = A_i f + \eta_i$, where $A_i = \mathcal{S}_\Omega M^{i-1}$ for $i \in \{1, \dots, L\}$, $f \in PW_{\lambda_k}$, $\mathcal{S}_\Omega$ is the sampling operator for some $\Omega \subseteq \{1, \dots, d\}$, and $\eta_i$ are i.i.d. random variables with a zero mean and a variance matrix $\sigma^2 I$, we have obtained a bound for the expected error of approximately reconstructing $f$ via a least squares solution which has been shown to be an improvement on a previously known bound in the case of no subsampling.**

## I. Introduction

The Dynamical Sampling problem was inspired from a paper by Lu and Vetterli where they sought to reconstruct a vector, using an insufficient spacial sampling density by employing an evolution operator to compensate for the reduced spacial sampling. In dynamical sampling, the goal is the same, however we no longer have a notion of domain for the vectors in $H$, some separable Hilbert space, and instead seek to recover any vector $f \in H$ using the samples $\langle A^n f, g\rangle_{n=0,1,\dots L(g), g\in G}$ where $G$ is a countable subset of $H$, and $L$ is a map from $G$ to $N \cup \{\infty\}$. Since the samples given above are equivalent to $\langle f, (A^*)^n g\rangle_{n=0,1,\dots L(g), g\in G}$ we find that any vector $f \in H$ can be stably reconstructed from the samples above if and only if $\{(A^*)^n g\}_{n=0,1,\dots L(g), g\in G}$ is a frame for $H$. Due to this, we forgo the adjoint and seek conditions for $A$, $G$, and $L$ such that the system $\{A^n g\}_{n=0,1,\dots L(g), g\in G}$ is a frame, Bessel, complete, minimal, etc.

## II. Main Results

*Theorem 2.1:* Let $M$ be the Laplacian of a graph with $d$ vertices. Assume that $M$ has distinct real ( and non-negative as $M$ is positive semidefinite) eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ and that the eigenvalues satisfy $\lambda_1 < \lambda_2 < \dots \lambda_n$. Suppose $PW_{\lambda_k} = span\{v_1, \dots, v_m\}$ (where the $v_i$ are taken to be mutually orthogonal as $M$ is self-adjoint and have norm 1) has dimension $m \le d$. Let $f \in PW_{\lambda_k}$ so that $f = [v_1 \dots v_m]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix}$. Let $x = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix}$. Let $\mathcal{S}_\Omega \in C^{dxd}$ be the sampling operator for some $\Omega \subseteq \{1, \dots, d\}$. Assume $\tilde{y}_i = A_i f + \eta_i$, where $\eta_i$ are i.i.d. random variables with a zero mean and a variance matrix $\sigma^2 I$ where $A_i = \mathcal{S}_\Omega M^{i-1}$ for $i \in \{1, \dots, L\}$. Suppose $\mathcal{A}_L = \begin{bmatrix}\mathcal{S}_\Omega I\\ \mathcal{S}_\Omega M\\ \vdots\\ \mathcal{S}_\Omega M^{L-1}\end{bmatrix}$ has full rank, then the matrix

$$\mathcal{A}_{(PW_{\lambda_k},L)} = \begin{bmatrix}\mathcal{S}_\Omega[v_1 \dots v_m]\\ \mathcal{S}_\Omega[v_1 \dots v_m]T\\ \vdots\\ \mathcal{S}_\Omega[v_1 \dots v_m]T^{L-1}\end{bmatrix} = \begin{bmatrix}\widehat{A_1}\\ \widehat{A_2}\\ \vdots\\ \widehat{A_L}\end{bmatrix}$$

(where $T$ is the $mxm$ leading principal submatrix of the diagonal matrix $D = \begin{bmatrix}\lambda_1 I_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n I_n\end{bmatrix}$ from the eigendecomposition $M = UDU^*$ where $U$ is unitary) has full rank, and $f$ can be approximately recovered via $\widehat{x_L} = \underset{z}{\text{argmin}} \sum_{i=1}^L \|\widehat{A_i} z - \tilde{y}_i\|_2^2$. Denote the error $\epsilon_L^{PW_{\lambda_k}} = \widehat{x_L} - x$. Then we have $E(\|\epsilon_L^{PW_{\lambda_k}}\|_2^2) \le \sigma^2 tr((\mathcal{A}^*_{(PW_{\lambda_k},L)}\mathcal{A}_{(PW_{\lambda_k},L)})^{-1}) = \sigma^2 \sum_{i=j}^m 1/\rho_j(L)$, where $\rho_j(L)$, $1 \le j \le m$, are the eigenvalues of the matrix $\mathcal{A}^*_{(PW_{\lambda_k},L)}\mathcal{A}_{(PW_{\lambda_k},L)}$.

*Remark 2.2:* Theorem 2.1, which uses Proposition 4.1 in "Dynamical Sampling with Additive Random Noise", is noteworthy since it employs the structure of the Paley-Wiener space to obtain another bound than the one previously known. This bound has been shown to be strictly less than the other in the case where there is no subsampling.

*Theorem 2.3:* Let $M$ be the Laplacian of a graph with $d$ vertices and $D$ the diagonal matrix from the eigendecomposition of $M$. Suppose for some vectors $b_i \in C^d$, $i \in \Omega \subset \{1, \dots, d\}$, we have $\{P_j(b_i) : i \in \Omega\}$ complete in $P_j(C^d)$ for each $1 \le j \le k$, where $P_j$ are orthogonal projections onto the span of the standard basis vectors corresponding to the position of $\lambda_j$ in $D$. Given $dim(PW_{\lambda_k}) = m \le d$, then for $\widehat{C^m} = \{\vec{x} \in C^d : x_i = 0\ \forall i \ge m+1\}$, there exists $A : \widehat{C^m} \to PW_{\lambda_k}$ such that $\{A((\sum_{j=1}^k P_j)D^l b_i) : l \in \{0, 1, \dots r_i - 1\}, i \in \Omega\}$ is a frame for $PW_{\lambda_k}$.

*Remark 2.4:* Theorem 2.3, which uses Theorem 2.2 in "Dynamical Sampling" is notable since it gives conditions for the reconstruction of Paley-Wiener spaces of graphs via dynamical samples.

## III. Conclusion

The term Dynamical Sampling classifies the set of inverse problems arising from considering samples of a signal and its future states under the action of a linear evolution operator. In Dynamical Sampling, both the signal, $f$, and the driving operator, $A$, may be unknown however by considering samples of the form $\{A^j f(i) : j = 0, \dots l_i, i \in \Omega\}$, we can obtain conditions that allow for reconstruction of the signal $f$ and/or the operator $A$. The framework of Dynamical Sampling has been applied to various areas and recently has been used in the graph theoretical setting.

## References

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