A Generative Variational Model for Inverse Problems in Imaging

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***Abstract*—We develop a generative model for solving inverse problems in imaging, where the unknown is generated from a multi-layer convolutions. The method is introduced and analyzed in function space. We prove theoretical results and show experiments for different imaging applications with comparisons to existing methods.**

I. Introduction

The following is a short version of our work [4].

We are concerned with the solution of inverse problems in imaging. That is, given $A : X \to Y$ a continuous operator between Banach spaces and (possibly noisy) data $y \in Y$, we want to solve $A(u) \approx y$ for u. For simplicity, we will restrict A to be linear. A typical approach to overcome ill-posedness in inverse problems are variational methods, where we consider solving

$$\min_u \lambda \mathcal{D}_y(Au) + \mathcal{R}(u), \quad \lambda > 0,$$

with $\mathcal{D}_y$ enforcing data fidelity and the regularization functional $\mathcal{R}$ ensuring well-posedness and acting as a prior (e.g. the total variation). In [5] the authors propose the deep image prior (DIP), i.e., they solve inverse problems using a generative convolutional neural network $g_\theta(z)$ with input (latent variable) z and parameters θ via computing

$$\hat{\theta} \in \operatorname*{argmin}_\theta \mathcal{D}_y(A g_\theta(z)), \tag{1}$$

where z is fixed. Afterwards the solution is obtained as $u = g_{\hat{\theta}}(z)$. Inspired by the remarkable experimental results of (DIP) we propose a novel variational regularization method where we combine the deep image prior with a second regularization via infimal convolution by solving

$$\min_{u,v} \lambda \mathcal{D}_y(Au) + \nu \mathcal{R}(u - v) + (1-\nu)\mathcal{G}(v). \tag{GR}$$

Here, $\mathcal{G}$ is a so-called generative prior capturing the network architecture. In this work, we introduce and analyze $\mathcal{G}$ in function space, which allows not only for a proof of well-posedness of the corresponding regularization approach, but also enables us to investigate the regularity of solutions.

II. The Generative Prior

For $q \in (1, 2]$, the generative prior $\mathcal{G} : L^q(\Omega) \to [0, \infty]$ is defined as

$$\mathcal{G}(v) = \inf_{\substack{\mu=(\mu^1,\dots\mu^L),\\ \theta=(\theta^1,\dots\theta^L)}} \|\mu\|_{\mathcal{M}} \quad \text{s.t.} \begin{cases} i) & v = \sum_{n=1}^{N_1} \mu_n^1 * \theta_{n,1}^1 \\ ii) & \mu_k^{l-1} = \sum_{n=1}^{N_l} \mu_n^l * \theta_{n,k}^l \\ iii) & \|\theta_{n,k}^l\|_2 \leq 1 \\ iv) & Z\theta = 0. \end{cases} \tag{2}$$

The latent variables μ are modeled as radon measures and the filter kernels θ as L^2 functions. Constraints $i)$ and $ii)$ in (2) ensure that v is generated from multiple consecutive convolutions. We penalize the radon norm of the latent variables on all layers which replaces non-linear activation functions in our model and acts as regularization. We bound the norm of the filter kernels in order to remove ambiguities arising from the bilinearity of the convolution. We allow for additional constraints on θ via the linear and bounded operator Z (the concrete choice of Z will depend on $\mathcal{R}$). No training is needed for our method.

III. Main Results

Theorem 3.1: (Regularity) If in the definition of $\mathcal{G}$, if the number of layers $L \geq 2$, then $\mathcal{G}(u) < \infty$ implies $u \in C(\bar{\Omega})$.

Hence, our generative prior cannot generate discontinuities, which indicates the capability of removing noise. This also means it cannot generate jump discontinuities, therefore we combine $\mathcal{G}$ with a second regularization $\mathcal{R}$ allowing for discontinuities (e.g. the total variation). Moreover, in the case $R = \mathrm{TV}$ we choose $Z\theta = (\int \theta_{n,1}^1 \ dx)_n$ motivated by the fact, that TV does not penalize constants.

Theorem 3.2: (Well-posedness) Under standard assumptions (see [4]) (GR) is well-posed. That is,

1) the problem (GR) admits a solution,
2) the solution is weakly subsequentially stable up to shifts of u in the intersection of $\ker(A)$ with the invariant space of $\mathcal{R}$ and
3) if the data converges to the ground truth $y_n \to y$, then, with an appropriate choice of parameters λ, the solutions $(u_n, v_n)_n$ converge (as in item 2) to a solution of $Au = y$ with minimal value of $\nu\mathcal{R}(u - v) + (1 - \nu)\mathcal{G}(v)$.

IV. Experiments

In Figures 1 to 4 we show practical results in comparison to a custom implementation of total generalized variation regularization (TGV)[2] and to (DIP) [6]. To reproduce the results see [3].

V. Conclusion

Data driven models, as is the generative prior, have shown to be very useful for the regularization of inverse problems [5, 1]. Yet, the level of mathematical foundation is still inferior compared to conventional variational methods. With the proposed work we have partly overcome this issue, by introducing and investigating a generative model in function space. Moreover, in experiments, our method performs at least equally well compared to recent deep learning methods (see [4]).

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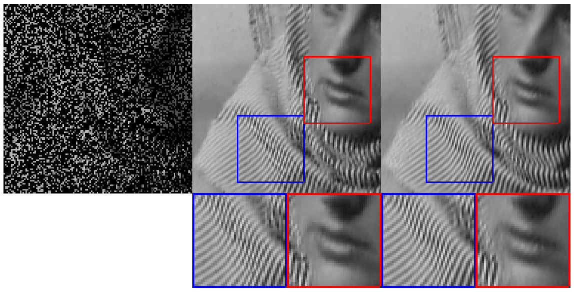

Fig. 1: Inpainting from 30% known pixels. From left to right: Data, DIP, proposed.

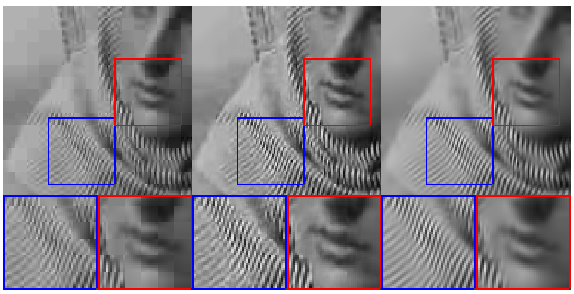

Fig. 2: JPEG decompression. From left to right: Data, DIP, proposed.

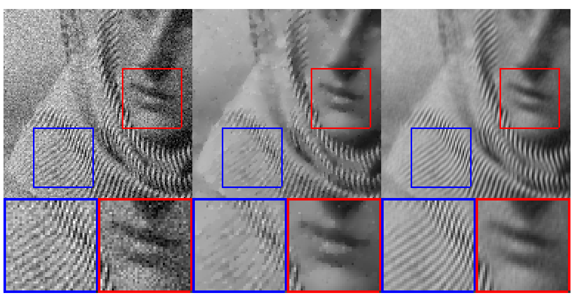

Fig. 3: Denoising. From left to right: Data, TGV, proposed.

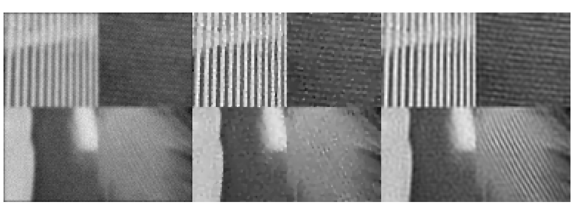

Fig. 4: Deconvolution. From left to right: Data, TGV, proposed.

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