Frames via Unilateral Iterations of Bounded Operators

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***Abstract*—Motivated by recent work [1, 2, 5] regarding frames given by iterations of bounded operators, we consider the case for frames obtained by iterations of a pair of commuting operators $T_1, T_2 \in B(H)$. We utilize results in [6, 7] on shift-invariant subspaces of $H^2(\mathcal{T}^2)$ to obtain necessary and sufficient conditions for a system $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$ to be a frame for H. Also, a result in [3] regarding operator representations of frames via bounded operators has been found which shows that the structure of a frame given by the orbit of a bounded operator must be reflected in a certain subspace of $\ell^2(N_0)$, and it can be shown that the inner functions mentioned in Theorem 3.6 in [5] and in [3] differ at most by a constant factor of modulus one.**

I. Introduction

In Theorem 3.6 of [5] it is shown that for any separable infinite-dimensional Hilbert Space, H, a system $\{T^n f_0\}_{n\in N_0} \subset H$, where $T \in B(H)$, is a frame if and only if there exists a model space, $K_\theta = H^2(\mathcal{T}) \ominus \theta H^2(\mathcal{T})$, where $\theta \in H^2(\mathcal{T})$ is an inner function, such that the operator T is similar to the compressed shift operator, S_θ, on K_θ and f_0 is the image under the map in the similarity relation of the projection onto K_θ of the function $1_{\mathcal{T}}$ (or in their notation, $(T, f_0) \cong (S_\theta, P_{K_\theta} 1_{\mathcal{T}})$). In this vein, we seek to determine necessary and sufficient conditions for a system $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$ to be a frame for H. In the space $H^2(\mathcal{T}^2) = \{f(z,w) \in L^2(\mathcal{T}^2) : \hat{f}(j,l) = \int_{\mathcal{T}^2} f(z,w)\overline{z}^j \overline{w}^l dm(z,w) = 0 \text{ if } j < 0 \text{ or } l < 0\ \}$, we say a subspace $M \subseteq H^2(\mathcal{T}^2)$ is shift-invariant if it is invariant under both shift operators S_1 and S_2. That is $S_1 M = zM \subset M$ and $S_2 M = wM \subset M$. By obtaining a similarity relation between the operators T_1, T_2 and a compression of the shift operators to an appropriate subspace of $H^2(\mathcal{T}^2)$, we can determine precisely when a system $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$ is a frame for H. Also, a necessary and sufficient condition in [3] regarding operator representations of frames via a single bounded operator has been found which shows that the structure of a frame given by the orbit of a bounded operator must be reflected in a certain subspace of $\ell^2(N_0)$, namely the kernel of the synthesis operator.

II. Main Results

A. First main result

Theorem 2.1: Let $\{T_1^i T_2^j f_0\}_{i,j\geq 0} \subset H$ where $T_1, T_2 \in B(H)$ commute. Let $R_1 \in B(\ell^2(N_0 \times N_0))$ be the right shift in the first component and $R_2 \in B(\ell^2(N_0 \times N_0))$ be the right shift in the second component. Assume R_1, R_2 doubly commute on the kernel of the synthesis operator associated with $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$. Then $\{T_1^i T_2^j f_0\}_{i,j\geq 0}$ is a frame if and only if there exists an invariant subspace $\theta H^2(\mathcal{T}^2)$, with $\theta(z,w)$ a unique inner function, such that $(T_1 T_2, f_0) \cong (S_{\theta_1} S_{\theta_2}, P_{K_\theta} 1_{\mathcal{T}^2})$ where P_{K_θ} is the orthogonal projection onto $K_\theta = H^2(\mathcal{T}^2) \ominus \theta H^2(\mathcal{T}^2)$ and $S_{\theta_1} = P_{K_\theta}\ S_1|_{K_\theta}$ and $S_{\theta_2} = P_{K_\theta}\ S_2|_{K_\theta}$.

Remark 2.2: Theorem 2.1, which uses a result in [6] regarding Beurling-type shift invariant subspaces of $H^2(\mathcal{T}^2)$, is notable since it provides necessary and sufficient conditions for a system generated by commuting operators in $B(H)$ to be a frame and it seamlessly extends the result in Theorem 3.6 of [5].

B. Second main result

Theorem 2.3: Let S be the right shift operator on $\ell^2(N_0)$. A frame can be given in the form $\{T^n f_0\}_{n\geq 0} \subset H$, with $T \in B(H)$, if and only the frame is a Riesz basis or the frame is linearly independent and the kernel of the synthesis operator of the frame is generated by the vectors $\{S^n c\}_{n\geq 0}$ for some fixed scalar sequence $c \in \ell^2(N_0)$ such that its image under $V : \ell^2(N_0) \to H^2$, where $V(c_0, c_1, \ldots) = \sum_{n\geq 0} c_n z^n$, is an inner function.

Remark 2.4: Theorem 2.3, which uses Beurling's Theorem [4] is noteworthy since it shows that H can be reconstructed by the orbit of a bounded operator on a single vector only when the kernel of the synthesis operator is generated by the orbit of a shift operator on a single scalar sequence. This result shows that the structure of a frame given by the orbit of a bounded operator must be reflected in a certain subspace of $\ell^2(N_0)$. We also show that the inner function given in this theorem differs at most by a constant factor of modulus one from the inner function in Theorem 3.6 of [5].

III. Conclusion

The Dynamical Sampling problem asks when a system generated by iterations of a bounded operator on vectors in H is a frame or has properties such as completeness, exactness, etc. This problem has prompted related questions such as when can a given frame be represented as the orbit of a bounded operator? Our current work seeks to extend recent results to obtain necessary and sufficient conditions for a system generated by a pair of commuting bounded operators to be a frame and to determine properties of frames that are generated by orbits of bounded operators.

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