Possible windows for finite-dimensional STFT phase retrieval

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***Abstract*—We study phase retrieval for the finite-dimensional short-time Fourier transform (STFT), i.e. the problem of recovering a signal $f \in \mathbb{C}^d$ (up to a global phase factor) from its spectogram $|V_g f|^2$ w.r.t. a given window $g \in \mathbb{C}^d$. It is well-known that this recovery is always possible, once the window's ambiguity function vanishes nowhere. We discuss whether this condition is also necessary for a window g to do phase retrieval. Additionally, we investigate the possibility of phase retrieval using real-valued windows.**

I. Introduction

Let $d \geq 2$ and consider a window $g \in \mathbb{C}^d$ as well as a signal $f \in \mathbb{C}^d$. The short-time Fourier transform (STFT) of f w.r.t. g is defined as

$$V_g f(k,l) = \sum_{j=0}^{d-1} f_j \overline{g_{j-k}} e^{-2\pi i j l/d} \qquad (0 \leq k, l \leq d-1),$$

where the indices are to be understood modulo d. For fixed g, *phase retrieval* tries to recover a signal f from the measurement $|V_g f|^2$. However, since the mapping $f \mapsto V_g f$ is linear, it is obvious that f and γf yield the same measurements for every $\gamma \in \mathbb{T} := \{z \in \mathbb{C} \ : \ |z| = 1\}$. Thus, we can only hope to recover f up to a global phase factor.

With this observation in mind, we say that the window g *does phase retrieval*, iff every $f \in \mathbb{C}^d$ can be recovered up to global phase, i.e. iff the map

$$\mathbb{C}^d/\mathbb{T} \to \mathbb{C}^{d \times d}, \quad [f] \mapsto |V_g f|^2$$

is injective. It has been shown (cf. [1]) that a window g does phase retrieval, provided

$$V_g g(k,l) \neq 0 \qquad \text{for all } 0 \leq k, l \leq d-1, \tag{1}$$

and that this condition is satisfied with probability 1 when the window is picked randomly (w.r.t. complex standard normal distribution). In practice however, one might be forced to use a fixed window g. In this situation, we ask whether condition (1) is also necessary for phase retrieval. Additionally, we discuss if g can be chosen to be real-valued. It turns out that the answers to both questions depend on the dimension d. The results in section II are part of the research described in the preprint [3].

II. Main Results

The main insight to both questions mentioned in the previous section is the relation

$$\widehat{|V_g f|^2}(k,l) = d \cdot V_f f(-l,k) \cdot \overline{V_g g(-l,k)}$$

(cf. e.g. [2]). This yields the observation that g does phase retrieval if and only if every $f \in \mathbb{C}^d$ can be recovered up to global phase from the sampled measurement $V_f f|_{\Omega(g)}$, where

$$\Omega(g) := \{0 \leq k, l \leq d-1 \ : \ V_g g(k,l) \neq 0\}.$$

Since it is essentially folklore that the full measurement $V_f f$ determines f uniquely (up to global phase), this is one way to prove (1) to be a sufficient condition for phase retrieval (as is done in [2]).

A. Phase retrieval with real-valued windows

In this subsection, we ask whether there exist real-valued windows doing phase retrieval. It turns out that the answer to this question solely depends on the parity of the dimension d.

Using the sufficiency of condition (1), one can easily find real-valued phase retrieval windows for odd dimensions.

Example 2.1: Let $d \geq 3$ be odd and consider $g \in \mathbb{R}^d$, given by

$$g_j := \begin{cases} 2^j, & \text{if } 0 \leq j \leq \frac{d-1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\Omega(g) = \{0, \ldots, d-1\}^2$ and consequently, g does phase retrieval.

On the other hand, it can be shown that in even dimensions, every real-valued g satisfies $V_g g(\frac{d}{2}, l) = 0$ for every odd $0 \leq l \leq d-1$, and that this is enough to cause non-trivial ambiguities in phase retrieval.

Theorem 2.2: Let d be even and consider $g \in \mathbb{R}^d$. Then, $f, \tilde{f} \in \mathbb{C}^d$, given by

$$f_j := \begin{cases} 1, & \text{if } j = 0, \\ 1+i, & \text{if } j = \frac{d}{2}, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{f}_j := \begin{cases} 1, & \text{if } j = 0, \\ 1-i, & \text{if } j = \frac{d}{2}, \\ 0, & \text{otherwise} \end{cases}$$

satisfy $|V_g f|^2 = \left|V_g \tilde{f}\right|^2$. In particular, g can't do phase retrieval.

Consequently, phase retrieval with real-valued windows is exclusive to odd dimensions.

B. Necessity of condition (1)

The fact whether (1) is not only sufficient but also necessary for phase retrieval, also depends on the dimension. However, in contrast to the previous subsection, there is no dependence on the parity of the dimension: Necessity only holds true, when d is small enough.

Theorem 2.3: Let $d \in \{2,3\}$. Then, g does phase retrieval if and only if (1) holds true.

For larger dimensions, we present two types of windows that do phase retrieval despite not satisfying (1). The first construction is restricted to even dimensions.

Example 2.4: Let $d \geq 4$ be even. Consider $g \in \mathbb{C}^d$, given by

$$g_j := \begin{cases} 2^j, & \text{if } 0 \leq j \leq \frac{d}{2}-1, \\ \frac{1}{2^{j-d/2}} \left(1 + \sqrt{2^{4j-d}-1} \cdot i\right), & \text{if } j \in \left\{\frac{d}{2}, \frac{d}{2}+1\right\}, \\ \frac{1}{2^{j-d/2}} \left(2 + \sqrt{2^{4j-d}-4} \cdot i\right), & \text{if } \frac{d}{2}+2 \leq j \leq d-1, \end{cases}$$

when d is a multiple of 4, and

$$g_j := \begin{cases} 2^j, & \text{if } 0 \leq j \leq \frac{d}{2}-1, \\ \frac{1}{2^{j-d/2}} \left(\sqrt{2^{4j-d}-1} + i\right), & \text{if } j \in \left\{\frac{d}{2}, \frac{d}{2}+1\right\}, \\ \frac{1}{2^{j-d/2}} \left(\sqrt{2^{4j-d}-4} + 2i\right), & \text{if } \frac{d}{2}+2 \leq j \leq d-1, \end{cases}$$

otherwise. Then, $V_g g(\frac{d}{2}, \frac{d}{2}) = 0$, but g does phase retrieval.

For odd dimensions, there are also phase retrieval windows satisfying $\Omega(g) \subsetneq \{0, \ldots, d-1\}^2$. However, the formulation (and proof) is less explicit than before.

Theorem 2.5: Let $d \geq 5$. Then, there exist $g \in \mathbb{C}^d$ and $0 \leq l \leq d-1$ such that

(i) g does phase retrieval,
(ii) $\operatorname{supp}(g) \subseteq \left\{0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor\right\}$ and
(iii) $V_g g(0,l) = V_g g(0,-l) = 0$.

Note that the theorem doesn't require d to be odd and that the examples obtained from it for even $d \geq 6$ are evidently distinct from those in Example 2.4.

III. Conclusion

First, we have shown that phase retrieval with real-valued windows is possible if and only if the dimension is odd. The examples for odd dimensions are fairly easy, whereas the proof of non-existence for even dimensions uses the special role of the shift $k = \frac{d}{2}$.

As for condition (1), we have seen that it fails to be a necessary condition for phase retrieval once $d \geq 4$. This is in line with the intuition that the loss of a single time-frequency atom $V_f f(k,l)$ should be more severe in lower dimensions than in higher ones. Note that many reconstruction procedures for phase retrieval involve the full row of measurements $V_f f(0,\cdot)$ to compute $|f|$ in a first step. Theorem 2.5 shows that knowledge of this full row is not necessary for phase retrieval. Finally, it is worth noting that Example 2.4 and Theorem 2.5 particularly imply examples of sufficient conditions for recovering a signal f from the sampled measurement $V_f f|_\Lambda$ on some sampling set $\Lambda \subseteq \{0, \ldots, d-1\}$.

References

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[3] D. Bartusel, *Injectivity conditions for STFT phase retrieval on $\mathbb{Z}$, $\mathbb{Z}_d$ and $\mathbb{R}^d$.* arXiv:2206.06729v1 [math.FA], June 2022.