Complementable and hierarchical Riesz bases of exponentials

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***Abstract*—We give partial answers to two questions concerning Riesz exponential bases. Firstly, we show that the standard Fourier basis is complementable with respect to any finite union of intervals. Secondly, we show that any finite union of intervals with endpoints independent over $\mathbb{Q}$ admits a so-called hierarchical Riesz exponential basis.**

I. Introduction

One of the fundamental questions of harmonic analysis is figuring out when, given some positive (but finite) measure subset $S \subset \mathbb{R}^d$, one can find a discrete set $\Lambda \subset \mathbb{R}$ such that the associated set of exponentials $E(\Lambda) := \{e^{2\pi i \lambda \cdot} : \lambda \in \Lambda\}$ forms a basis for $L^2(S)$.

Extensive work around the Fuglede conjecture has shown that looking for orthonormal bases of exponentials is very restrictive. For example, Iosevich, Katz and Tao proved that if $S \subset \mathbb{R}^2$ is convex, it admits a orthogonal basis of exponentials if and only if S tiles the plane by translations. A few years before, Iosevich, Katz and Pedersen had shown that the unit ball in dimension $d > 1$ admits no ONB of exponentials.

The most natural generalization of the notion of an ONB is that of a Riesz basis. Namely, $E(\Lambda)$ forms a Riesz basis of exponentials for $L^2(S)$ if it is complete and if there exist constants $0 < A \le B$ such that for any sequence $c \in \ell_2(\Lambda)$ the following inequality holds:

$$A\|c\|_{\ell_2}^2 \le \|\sum_{\lambda\in\Lambda} c_\lambda e^{2\pi i\lambda\cdot}\|_{L^2(S)}^2 \le B\|c\|_{\ell_2}^2$$

.

The question whether any bounded subset of $\mathbb{R}$ admits a Riesz exponential basis has only been answered in the last year. Kozma, Nitzan and Olevskii explicitly constructed by an iterated procedure a set included in $[0,1] \cap [2,3]$ with no such basis.

On the other hand, there are numerous results about the existence of Riesz exponential bases for more regular sets, especially for intervals and finite union of intervals. Seip proved that any interval contained in $[0,1]$ admits an exponential Riesz basis with integer frequencies. More recently, a significant breakthrough was achieved by Kozma and Nitzan, who showed that for any finite union of disjoint intervals $\cup_{k=1}^n [a_k, b_k] \subset [0,1]$, one can find some $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2(\cup_{k=1}^n [a_k, b_k])$.

Pfander, Revay and Walnut showed that if the intervals $[a_k, b_k]$ actually partition $[0,1]$, one can partition $\mathbb{Z} = \cup_{k=1}^n \Lambda_k$ such that for any consecutive index set $J \subset \{1, 2, \ldots, n\}$, $E(\cup_{k\in J}\Lambda_k)$ is a Riesz basis for $L^2(\cup_{k\in J}[a_k, b_k])$.

These two recent papers form the foundation of our investigation and motivate the following two problems.

Problem 1 (Complementability of exponential Riesz bases). Let $S \subset \mathbb{R}$ have finite positive measure and let $\Lambda \subset \mathbb{R}$ be discrete such that $E(\Lambda)$ is a Riesz basis for $L^2(S)$. Under what conditions is the pair (S, Λ) complementable? In other words, given an arbitrary positive but finite measure set S' disjoint with S, can one find a discrete $\Lambda' \subset \mathbb{R}$ such that $E(\Lambda')$ is a Riesz basis for $L^2(S')$ and $E(\Lambda \cup \Lambda')$ is a Riesz basis for $L^2(S \cup S')$?

Problem 2 (Hierarchical integer frequency Riesz bases). Given a finite family of disjoint positive measure sets $S_1, S_2, \ldots, S_n \subset [0,1]$, can one find disjoint integer sets $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \subset \mathbb{Z}$ such that for *any* index set $J \subset \{1, 2, \ldots, n\}$ we have that $E(\cup_{k\in J}\Lambda_k)$ is a Riesz basis for $L^2(\cup_{k\in J} S_k)$?

The main technical tool used for our investigation is a key lemma of Kozma and Nitzan.

Lemma 1.1: Let $S \subset [0,1]$ be a measurable set and let $N \in \mathbb{N}$. For arbitrary $n \in \{1, 2, \ldots, N\}$, define the auxiliary sets

$$A_{\ge n} := \{t \in [0, \tfrac{1}{N}] : t + \tfrac{k}{N} \in S \text{ for at least } n \text{ values of } 0 \le k \le N-1\}.$$

If there exist sets $\Lambda_1, \Lambda_2, \ldots, \Lambda_N \subset N\mathbb{Z}$ such that $E(\Lambda_n)$ is a Riesz basis for $L^2(A_{\ge n})$ for all $1 \le n \le N$, then $E(\cup_{n=1}^N (\Lambda_n + n))$ is a Riesz basis for $L^2(S)$.

II. Main Results

We provide positive answers to *Problem 1* and *Problem 2* in the case of all sets S involved being intervals or finite unions of intervals under some technical conditions.

A. Complementability

Theorem 2.1: The standard Fourier basis of $L^2([0,1])$ is complementable with respect to any finite union of bounded intervals. In other words, given an arbitrary collection of L intervals $[a_\ell, b_\ell]$ $(1 \le \ell \le L)$ such that $1 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_L < b_L \le N$, there exists a set $\Lambda' \subset (\frac{1}{N}\mathbb{Z}) \setminus \mathbb{Z}$ such that $E(\Lambda')$ is a Riesz basis for $L^2(\cup_{\ell=1}^L [a_\ell, b_\ell])$ and $E(\mathbb{Z} \cup \Lambda')$ is a Riesz basis for $L^2([0,1] \cup [a_1, b_1] \cup \cdots \cup [a_L, b_L])$.

Remark 2.2: This is a corollary of a dilated version of Lemma 1.1 by keeping track of the endpoints a_ℓ, b_ℓ modulo $\frac{1}{N}$.

B. Hierarchical bases

Theorem 2.3: Let $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_L < b_L < 1$. If the numbers $1, a_1, \ldots, a_L, b_1, \ldots, b_L$ are linearly independent over $\mathbb{Q}$, there exist disjoint integer frequency sets $\Lambda_\ell \subset \mathbb{Z}$ such that, for *any* index set $J \subset \{1, 2, \ldots, L\}$, the system $E(\cup_{\ell\in J}\Lambda_\ell)$ is a Riesz basis for $L^2(\cup_{\ell\in J}[a_\ell, b_\ell])$.

Remark 2.4: This result hinges on two observations.

Firstly, due to Chebotarëv's theorem on roots of unity, in the case when we choose a window $[0, \frac{1}{N}]$ for constructing the auxiliary sets $A_{\ge n}$ with N prime, we can replace the shift factors $\Lambda_n + n$ to $\Lambda_n + \pi(n)$ for arbitrary permutations $\pi : \{1, \ldots, N\} \to \{1, \ldots, N\}$.

Secondly, the Kronecker-Weyl equidistribution holds along primes. If the numbers $1, x_1, x_2, \ldots, x_d$ are linearly independent over $\mathbb{Q}$, the d-dimensional vectors $(\{px_1\}, \{px_2\}, \ldots, \{px_d\})$ are uniformly distributed in $[0,1]^d$ as p runs through the list of primes. This allows us to reorder the endpoints a_ℓ, b_ℓ arbitrarily modulo $\frac{1}{N}$ as long as N is a prime.

III. Conclusion

The answers to *Problem 1* and *Problem 2* for general sets S remain open. Specifically for the hierarchical question concerning intervals, we believe that linear independence over $\mathbb{Q}$ condition is a proof artifact and can probably be lifted with a novel approach. We conjecture the following.

Conjecture. Let $L \in \mathbb{N}$ and let the intervals $I_1, \dots, I_L$ partition (with repetitions allowed only on the endpoints) the interval $[0, 1]$. Then there exists a partition $\cup_{\ell=1}^{L} \Lambda_\ell = \mathbb{Z}$ such that, for *any* index set $J \subset \{1, 2, \dots, L\}$, we have that $E(\cup_{\ell \in J} \Lambda_\ell)$ is an exponential Riesz basis for $L^2(\cup_{\ell \in J} I_\ell)$.

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