

Sample complexity of ill-posed inverse problems via compressed sensing

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I. INTRODUCTION

Compressed sensing (CS) is one of the most thriving branches of applied mathematics in recent years, originating from the pioneering articles [5], [6]. The focus of CS is the recovery of sparse signals, namely those which can be expressed as a linear combination of a small number of vectors from a reference dictionary. The key principle of CS is that such sparsity prior knowledge can be exploited to drastically reduce the number of (random) samples that are needed to achieve full reconstruction of sparse signals – which actually constitute the most abundant species in real-world scenarios. The theoretical and algorithmic frameworks of CS are currently well understood for finite-dimensional vector spaces [7], [8]. Nevertheless, models arising from biomedical imaging and inverse problems for PDEs have inherently infinite-dimensional features, as they typically involve operators between Hilbert spaces or the reconstruction of a function from a finite number of pointwise samples. Further generalizations are possibly required, including frames and weights to model anisotropic measurements, or refined sparsity patterns. Several examples are known where discretization schemes introduce severe truncation errors and come along with a loss of the fine structure of the model, hence justifying the need of a purely infinite-dimensional theory of sampling. The theory of CS in Hilbert spaces has benefited from significant accomplishments over the last years, especially for problems modelled by isometric or unitary operators [2], [4]. Having in mind typical real-world scenarios with limited control on the geometry of the measurement procedures, these assumptions happen to be quite restrictive. Non-unitary models have been little explored so far [1], [3] and non-trivial modifications to the standard machinery are needed in such setting.

II. A UNIFYING ABSTRACT FRAMEWORK

The purpose of our contribution is to move the first steps towards a unified framework for the study of the sample complexity of several classes of ill-posed inverse problems, hence enforcing stable reconstruction of input signals (or suitable finite-dimensional approximations) from a finite number of noisy samples of its images. Our general framework encompasses both the case where the measurements are inner products with elements of a certain dictionary as well as the interpolation scenario where pointwise samples of a function are taken into account. We prove uniform recovery guarantees for bounded operators satisfying weak regularity assumptions (such as restricted injectivity or approximate diagonalization) which happen to be satisfied by many of the relevant operators in inverse problems.

To this aim, we first elaborate on a generalization of the restricted isometry property (g-RIP), that is a central notion in CS. Roughly speaking, in its basic form this condition guarantees that the measurement operators act approximately like an isometry on sparse signals. We substantially rearrange this notion in order to accommodate some typical features of inverse problems (e.g., lack of unitarity, the occurrence of weights and possible Tikhonov-like

regularization). While this condition ultimately suffices to obtain stable recovery guarantees, the main difficulty in this approach is to design a random sampling pattern resulting in a limited measurement operator that satisfies the g-RIP. The aforementioned interactions with regularization techniques are particularly useful, as they allow one to further relax the regularity assumptions on the model. For instance, any bounded compact operator can be taken into account – possibly at the price of slightly worse bounds, while a Tikhonov-like regularization further reduces the least number of samples needed to ensure stable reconstruction (sample complexity).

In general, both the sample complexity and the recovery estimates will depend on the norm of the inverse of a regularized version of the forward map. In a simplified form, we can prove lower bounds for the number m of samples such as

$$m \geq B \|G^{-1}\|^2 \kappa(G)^2 s \log^3 s \log n, \quad (1)$$

where B is a constant related to bounds on measurements, s is the weighted sparsity, n is an appropriate truncation bandwidth, $G \in \mathbb{C}^{n \times n}$ encodes the properties of the suitably truncated and regularized forward map and $\kappa(G)$ is its condition number. In particular, under a regularization parameter $\alpha \geq 0$, the matrix G is defined as

$$G = \sqrt{P_n F^* F \iota_n + \alpha^2 I_n}, \quad (2)$$

where $F: \ell^2(\mathbb{N}) \rightarrow \mathcal{H}$ is the forward map of the model, taking values in a Hilbert space \mathcal{H} , $\iota_n: \mathbb{C}^n \rightarrow \ell^2(\mathbb{N})$ and $P_n: \ell^2(\mathbb{N}) \rightarrow \mathbb{C}^n$ are, respectively, the inclusion and projection operators on the first n components. A given signal $x^\dagger \in \ell^2(\mathbb{N})$ can thus be recovered by solving the following weighted ℓ^1 -minimization problem:

$$\min_{x \in \mathbb{C}^n} \|x\|_{1,\omega}: \|Ax - y\|_2 \leq \eta, \quad (3)$$

where $\|x\|_{1,\omega} = \sum_i |x_i| \omega_i$ is the ℓ^1 -weighted norm, A is the sampling operator, $y = Ax^\dagger + \varepsilon$ is a vector of noisy samples of x^\dagger and η is the accuracy, which depends on the noise on the samples, on the norm of the input, on the truncation and the regularization parameters. If $\hat{x} \in \mathbb{C}^n$ denotes a solution of the minimization task, then

$$\|P_n x^\dagger - \hat{x}\|_2 \lesssim \frac{\sigma_s(P_n x^\dagger)_{1,\omega}}{\sqrt{s}} + \|G^{-1}\| \eta, \quad (4)$$

where $\sigma_s(P_n x^\dagger)_{1,\omega}$ is the error of best weighted s -sparse approximation of the n -truncated signal $P_n x^\dagger$. Precisely, explicit bounds for $\|G^{-1}\|$ are available in several cases of interests, such as those discussed below. We also exploit some indirect and peculiar invariance properties of the g-RIP.

Furthermore, we investigate the interplay between the sampling pattern, the sampling complexity and the quality of the recovery. Indeed, the choice of the probability distribution satisfying the g-RIP exhibits a trade-off between the available number of measurements and the desired resolution in the reconstruction. Depending on the features of the forward map, the matrix G can be generalized to

entail the distortion induced on the sampling system by the forward map of the model.

III. APPLICATIONS TO INVERSE PROBLEMS

We focus in particular on two classical models to illustrate the scope of our general framework.

The first example is an ill-posed variation of a standard problem in CS, that is the reconstruction of wavelet coefficients from noisy samples in the Fourier basis. We introduce a moderate source of ill-posedness by means of a frequency decay filter, motivated by the archetypical problem of inversion of (fractional) differentiation. More precisely, we consider a forward operator V on $L^2([0, 1])$ that acts on the elements of the (rearranged) Fourier basis $\{\psi_k(t) = e^{2\pi ikt}\}_{k \in \mathbb{N}}$ by

$$V\psi_k := \frac{1}{k^b} \psi_k, \quad k \in \mathbb{N}, \quad (5)$$

for a fixed $b > 1/2$. Thanks to a suitable balancing property we are able to handle the delicate transition to a finite-dimensional reconstruction problem with precise recovery estimates. For instance, given a target signal f^\dagger in the Sobolev space $H^a([0, 1])$ and the corresponding coefficients x^\dagger with respect to a basis of suitable wavelets, the following recovery guarantee holds:

$$\|P_n x^\dagger - \widehat{x}\|_2 \lesssim \frac{\sigma_s(P_n x^\dagger)_{1,\omega}}{\sqrt{s}} + \eta^{1-b/a}. \quad (6)$$

The second example that is discussed in detail is a deconvolution problem given pointwise samples of the filtered signal, motivated by the well-known quest for deblurring in signal analysis. In particular, we consider the following convolution operator:

$$Vf := f * \kappa_p, \quad \widehat{\kappa_p}(\omega) := (1 + |\omega|^2)^{-p/2}, \quad (7)$$

where κ_p is the Bessel potential of order $p > 1/2$ and f has compact support on \mathbb{R} . In this case, we are able to obtain analogous L^2 -recovery estimates for the signal of interest f^\dagger . A fundamental property which enters the picture is the quasi-diagonalization of the operator V with respect to a dictionary of compactly supported wavelets $(\phi_{j,k})_{j,k}$, namely

$$\|Vf\|_{L^2}^2 \asymp \sum_{j,k} w_j^{-2} |\langle f, \phi_{j,k} \rangle_{L^2}|^2 \quad (8)$$

for some $w_j \geq 1$. Moreover, there are $d_j \geq 1$, $j \in \mathbb{N}$, such that

$$\|\phi_{j,k} * \kappa_p\|_{L^\infty} \leq B \frac{1}{d_j}. \quad (9)$$

This leads to the notion of relative coherence, which plays a key role in the sample complexity of this model. Precisely, given a truncation bandwidth with maximum resolution 2^{j_0} , the relative coherence of the system is defined as

$$\max_{j \leq j_0} \frac{w_j}{d_j}. \quad (10)$$

The reconstruction is performed by taking random pointwise samples with respect to a certain probability distribution on a finite interval $[-K, K]$, where K depends on the maximum resolution 2^{j_0} .

This interpolation setting is particularly promising because it can be generalized to many other interesting cases and paves the way to more challenging problems such as the study of the sample complexity of the Radon transform.

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