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Unions of exponential Riesz bases

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***Abstract*—We develop new methods for combining exponential Riesz bases, which rely on directly taking unions of frequency sets and domains respectively.**

I. Introduction

For a discrete set $\Lambda \subset \mathbb{R}$ and a positive measure set $S \subset \mathbb{R}$, we consider the *exponential system* $E(\Lambda) := \{e^{2\pi i \lambda (\cdot)} : \lambda \in \Lambda\}$ on S. The set Λ is often called a *frequency set* or a *spectrum*, and S is called a *domain*.

Despite the simple formulation of exponential systems, the existence of exponential Riesz bases for a given domain S is a highly nontrivial problem. Up to date, there are only a handful of classes of domains for which exponential Riesz bases are known to exist, see e.g., [1], [2], [3], [4], [5], [6], [9], [10]. Recently, it was shown in [7] that there exists a bounded measurable set $S \subset \mathbb{R}$ such that no system of exponentials can be a Riesz basis for $L^2(S)$.

Aside from the question of existence of exponential Riesz bases, it is natural to ask whether the frequency sets and domains of (already given) exponential Riesz bases can be combined/split in a canonical way.

Question 1 (Combining exponential Riesz bases)**.** *Let $S_1, S_2 \subset \mathbb{R}$ be disjoint measurable sets. Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ be discrete sets with $\mathrm{dist}(\Lambda_1, \Lambda_2) := \inf\{|\lambda_1 - \lambda_2| : \lambda_1 \in \Lambda_1,\ \lambda_2 \in \Lambda_2\} > 0$, such that for each $n = 1, 2$, the system $E(\Lambda_n)$ is a Riesz basis for $L^2(S_n)$. Is the system $E(\Lambda_1 \cup \Lambda_2)$ a Riesz basis for $L^2(S_1 \cup S_2)$?*

Note that the condition $\mathrm{dist}(\Lambda_1, \Lambda_2) > 0$ is necessary, since $E(\Lambda_1 \cup \Lambda_2)$ is not even a Riesz sequence if $\mathrm{dist}(\Lambda_1, \Lambda_2) = 0$.

Question 2 (Splitting exponential Riesz bases)**.** *Let $S_1, S_2 \subset \mathbb{R}$ be disjoint measurable sets. If $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ are disjoint sets such that $E(\Lambda_1)$ and $E(\Lambda_1 \cup \Lambda_2)$ are Riesz bases for $L^2(S_1)$ and $L^2(S_1 \cup S_2)$ respectively, then is the system $E(\Lambda_2)$ a Riesz basis for $L^2(S_2)$?*

Unfortunately, both questions turn out negative.

For Question 1, note that $E(4\mathbb{Z} \cup (4\mathbb{Z}{+}2))$ is *not* a Riesz basis for $L^2([0, \frac{1}{4}) \cup [\frac{2}{4}, \frac{3}{4}))$, even though $E(4\mathbb{Z})$ and $E(4\mathbb{Z}{+}2)$ form Riesz bases for $L^2[0, \frac{1}{4})$ and $L^2[\frac{2}{4}, \frac{3}{4})$ respectively. However, this can be avoided by shifting one of the frequency sets. Indeed, for every $0 < \delta < 2$, the system $E(4\mathbb{Z} \cup (4\mathbb{Z}{+}2{+}\delta))$ is a Riesz basis for $L^2([0, \frac{1}{4}) \cup [\frac{2}{4}, \frac{3}{4}))$.

On the other hand, there are some cases where such shiftings do not help. For any $0 < \epsilon < \frac{1}{4}$ fixed, the sets

$$\begin{aligned} \Lambda^{(1)} &:= \{\ldots, -6, -4, -2, 0, 1{+}\epsilon, 3{+}\epsilon, 5{+}\epsilon, \ldots\}, \\ \Lambda^{(2)} &:= -\Lambda^{(1)} = \{\ldots, -5{-}\epsilon, -3{-}\epsilon, -1{-}\epsilon, 0, 2, 4, 6, \ldots\} \end{aligned} \tag{1}$$

satisfy that $E(\Lambda^{(1)})$, $E(\Lambda^{(2)})$, $E(\Lambda^{(1)} \cup \Lambda^{(2)})$ are Riesz bases for $L^2[0, 1/2)$, $L^2[1/2, 1)$, $L^2[0, 1)$, respectively. It is important to note that $\Lambda^{(1)}$ and $\Lambda^{(2)}$ have nonempty intersection $\Lambda^{(1)} \cap \Lambda^{(2)} = \{0\}$. For every $0 < \delta < \epsilon\ (< \frac{1}{4})$, we have $\mathrm{dist}(\Lambda^{(1)}, \Lambda^{(2)}{+}\delta) = \delta > 0$ but the system $E(\Lambda^{(1)} \cup (\Lambda^{(2)}{+}\delta))$ is *not* a Riesz basis for $L^2[0, 1)$.

Concerning Question 2, the sets $\Lambda_1 := \Lambda^{(1)}$ and $\Lambda_2 := \Lambda^{(2)} \backslash \{0\}$ are disjoint and satisfy that $E(\Lambda_1)$ and $E(\Lambda_1 \cup \Lambda_2)$ are Riesz bases for $L^2[0, 1/2)$ and $L^2[0, 1)$ respectively, but $E(\Lambda_2)$ is incomplete in $L^2[1/2, 1)$.

In this work, we develop new methods for combining exponential Riesz bases, which rely on directly taking unions of frequency sets and domains respectively.

II. Main Results

Theorem 1. *Let $S_1, S_2 \subset \mathbb{R}$ be disjoint bounded measurable sets with $S_1{+}a \subset S_2$ for some $a \in \mathbb{R}\backslash\{0\}$. If there exist sets $\Lambda_1 \subset \mathbb{R}\backslash\frac{1}{a}\mathbb{Z}$ and $\Lambda_2 \subset \frac{1}{a}\mathbb{Z}$ with $\mathrm{dist}(\Lambda_1, \frac{1}{a}\mathbb{Z}) > 0$ such that for each $\ell = 1, 2$, the system $E(\Lambda_\ell)$ is a Riesz basis (resp. a complete sequence, a Riesz sequence) for $L^2(S_\ell)$, then $E(\Lambda_1 \cup \Lambda_2)$ is a Riesz basis (resp. a complete sequence, a Riesz sequence) for $L^2(S_1 \cup S_2)$.*

Note that Theorem 1 with parameter $a = \frac{K}{N}$ for some $K, N \in \mathbb{N}$, would require $\Lambda_2 \subset \frac{N}{K}\mathbb{Z} = \cup_{k=0}^{K-1}(N\mathbb{Z}{+}k\frac{N}{K})$. For usage in applications, it is desirable to weaken the condition to $\Lambda_2 \subset \cup_{k=1}^{K}(N\mathbb{Z}{+}c_k)$ with *arbitrary* real numbers $c_1, c_2, \ldots, c_K \in [0, N)$. This can be achieved by assuming $S_1{+}\frac{k}{N} \subset S_2$ for $k = 1, 2, \ldots, K$, a condition which is indeed stronger than $S_1{+}\frac{K}{N} \subset S_2$.

Theorem 2. *Let $S_1, S_2 \subset \mathbb{R}$ be disjoint bounded measurable sets such that there exist numbers $K, N \in \mathbb{N}$ satisfying $S_1{+}\frac{k}{N} \subset S_2$ for $k = 1, 2, \ldots, K$. Let $c_1, c_2, \ldots, c_K \in [0, N)$ be distinct real numbers, and assume that there exist sets $\Lambda_1 \subset \mathbb{R}\backslash \cup_{k=1}^{K} (N\mathbb{Z}{+}c_k)$ and $\Lambda_2 \subset \cup_{k=1}^{K}(N\mathbb{Z}{+}c_k)$ with $\mathrm{dist}(\Lambda_1, \cup_{k=1}^{K}(N\mathbb{Z}{+}c_k)) > 0$, such that for each $\ell = 1, 2$, the system $E(\Lambda_\ell)$ is a Riesz basis (resp. a complete sequence, a Riesz sequence) for $L^2(S_\ell)$. Then $E(\Lambda_1 \cup \Lambda_2)$ is a Riesz basis (resp. a complete sequence, a Riesz sequence) for $L^2(S_1 \cup S_2)$.*

As a partial converse of Theorem 1 and Theorem 2, we have:

Theorem 3. *(a) Let $S_1, S_2 \subset \mathbb{R}$ be disjoint bounded measurable sets with $S_1{+}a \subset S_2$ for some $a \in \mathbb{R}\backslash\{0\}$. If there exist sets $\Lambda_1 \subset \mathbb{R}\backslash\frac{1}{a}\mathbb{Z}$ and $\Lambda_2 \subset \frac{1}{a}\mathbb{Z}$ such that $E(\Lambda_1 \cup \Lambda_2)$ is complete in $L^2(S_1 \cup S_2)$. Then $E(\Lambda_1)$ is complete in $L^2(S_1)$.*
(b) Let $S_1, S_2 \subset \mathbb{R}$ be disjoint bounded measurable sets such that there exist numbers $K, N \in \mathbb{N}$ satisfying $S_1{+}\frac{k}{N} \subset S_2$ for $k = 1, 2, \ldots, K$. Let $c_1, c_2, \ldots, c_K \in [0, N)$ be distinct real numbers, and assume that there exist sets $\Lambda_1 \subset \mathbb{R}\backslash\cup_{k=1}^{K} (N\mathbb{Z}{+}c_k)$ and $\Lambda_2 \subset \cup_{k=1}^{K}(N\mathbb{Z}{+}c_k)$ such that $E(\Lambda_1 \cup \Lambda_2)$ is complete in $L^2(S_1 \cup S_2)$. Then $E(\Lambda_1)$ is complete in $L^2(S_1)$.

As an application of Theorem 3, we have the following. Let $S_1 = [0, \frac{1}{2})$ and $S_2 = [\frac{1}{2}, 1)$. If there is a set $\Lambda_1 \subset \mathbb{R}\backslash 2\mathbb{Z}$ such that $E(\Lambda_1 \cup 2\mathbb{Z})$ is a Riesz basis for $L^2(S_1 \cup S_2)$, then $E(\Lambda_1)$ is necessarily complete in $L^2(S_1)$ by Theorem 3(a). This implies that there is no $\Lambda^{(1)} \subset \mathbb{R}$ with $\Lambda^{(1)} \cap 2\mathbb{Z} \neq \emptyset$ such that $E(\Lambda^{(1)})$ and $E(\Lambda^{(1)} \cup 2\mathbb{Z})$ are Riesz bases for $L^2(S_1)$ and $L^2(S_1 \cup S_2)$ respectively; hence, a pathological situation like (1) cannot happen.

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