On convergence of Ptychographic Iterative Engine for STFT phase retrieval

Oleh Melnyk
Helmholtz Center Munich,
TU Munich
oleh.melnyk@tum.de

***Abstract*—We consider a recovery from ptychographic measurements also known as the Short-Time Fourier Transform (STFT) phase retrieval. That is the unknown object of interest has to be reconstructed from as a set of diffraction patterns resulting from a series of localized illuminations. In this paper we study Ptychographic Iterative Engine (PIE), a popular iterative algorithm among practitioners, which uses the measurements corresponding to a single illumination at the time. We show that PIE is the stochastic gradient descent applied to the amplitude-based squared loss and derive its sublinear convergence guarantees.**

I. Introduction

Ptychography [1] is a lenseless imaging technique, which aims to reconstruct the object of interest from a set of diffraction patterns. For each pattern, a certain region of the specimen is illuminated with localized light and the resulting exit wave is then captured by a detector placed in the far-field. The applications of ptychography include biology [2], material science [3], crystallography [4] and many more.

For a mathematical description of the ptychographic experiment, let us denote by $x \in \mathbb{C}^d$ the object and by $w \in \mathbb{C}^d$ the illumination also known as window. Then, measurements are given by

$$y_j^r = |(F \operatorname{diag}(w) S_{-r} x)_j| + n_j^r, \quad (1)$$

for $j \in [d] := \{0, 1, ..., d-1\}$ corresponding to the j-th pixel of the detector and $r \in \mathcal{R} \subseteq [d]$ being the illumination region identifier. Vectors $n^r \in \mathbb{R}^d$ denote the noise, the matrix $F \in \mathbb{C}^{d \times d}$ is the Discrete Fourier Transform matrix, $\operatorname{diag}(w)$ is a diagonal matrix with entries of w on the main diagonal and S_{-r}, $r \in [d]$ is a family of the circular shift operators such that $(S_r v)_j = v_{j-r \bmod d}$ for any $v \in \mathbb{C}^d$.

By combining the matrices $A_r := F \operatorname{diag}(w) S_{-r}$ as a block row matrix A and y^r, n^r as long vectors y, n, we observe that the measurements (1) are a special case of the phase retrieval measurements

$$y = |Ax| + n.$$

Note that the absolute value $|\cdot|$ and other functions in this paper are applied entrywise. If $\mathcal{R} = [d]$, the matrix A corresponds to STFT with window w and otherwise it is a subsampled STFT matrix.

For the reconstruction from the measurements (1), we consider the PIE algorithm [5], which starts with an initial guess $z^0 \in \mathbb{C}^d$ and for $t \in \mathbb{N}$ constructs the t-th iterate by performing the next steps.

1) Select $r^t \in \mathcal{R}$.
2) Construct an exit wave $\psi = \operatorname{diag}(w) S_{-r^t} z^{t-1}$.
3) Compute its Fourier transform $\Psi = F\psi$.
4) Correct the magnitudes of Ψ as $\Psi' = \operatorname{diag}(\operatorname{sgn} \Psi) y^{r^t}$ with $\operatorname{sgn}(\alpha) = \alpha/|\alpha|$ if $\alpha \neq 0$ and $\operatorname{sgn}(0) = 0$.
5) Find an exit wave ψ' corresponding to Ψ' via $\psi' = F^{-1}\Psi'$.
6) Construct $z^t = z^{t-1} + \frac{\alpha}{\|w\|_\infty^2} S_{r^t} \operatorname{diag}(w)^* [\psi' - \psi]$,

where M^* and M^{-1} are the conjugate transpose and the inverse of matrix M, respectively, and $\|v\|_p$ is the ℓ_p-norm of the vector $v \in \mathbb{C}^d$, $1 \leq p \leq \infty$.

Originally in [5] the index r^t was selected such that illuminations corresponding to r^{t-1} and r^t overlap. Later in [6], [7] indices r^t are looping through the set $\mathcal{R}$, which is randomly shuffled every loop. In this paper we assume that r^t is selected uniformly at random from $\mathcal{R}$.

Despite the popularity of PIE and its alternations [6], [8], [9], [7], the convergence of these algorithms have not been established yet.

II. Main Results

Some of the algorithms for phase retrieval [10], [11], [12], [13], [14] perform the gradient based minimization of the amplitude-based squared loss function

$$\mathcal{L}_2(z; A) := \| |Az| - y \|_2^2, \quad (2)$$

with the generalized Wirtinger gradient defined as

$$\nabla \mathcal{L}_2(z) := A^*(Az - \operatorname{diag}(\operatorname{sgn} Az) y).$$

In the case of ptychography, the loss function (2) naturally splits into a sum $\mathcal{L}_2(z; A) = \sum_{r \in \mathcal{R}} \mathcal{L}_2(z; A_r)$, which allows to establish an interpretation of the PIE algorithm as stochastic gradient descent.

Theorem II.1. *The iteration of PIE is given by*

$$z^t = z^{t-1} - \frac{\alpha}{d\|w\|_\infty^2} \nabla \mathcal{L}_2(z^{t-1}; A_{r^t}). \quad (3)$$

The representation (3) allows to apply the modern theory on convergence of stochastic gradient descent [15].

Theorem II.2. *Let $\{z^t\}_{t \geq 0}$ be a sequence determined by PIE with an arbitrary starting point $z^0 \in \mathbb{C}^d$ and fix $\gamma > 0$. Denote by $|\mathcal{R}|$ the cardinality of the set $\mathcal{R}$. If the number of iterations T satisfies*

$$T \geq 4d^2 \gamma^{-4} |\mathcal{R}| \, \|w\|_\infty^2 \left\| \sum_{r \in \mathcal{R}} |S_r w|^2 \right\|_\infty \mathcal{L}_2^2(z^0),$$

and the parameter α satisfies

$$\alpha \leq \|w\|_\infty |\mathcal{R}|^{1/2} T^{-1/2} \left\| \sum_{r \in \mathcal{R}} |S_r w|^2 \right\|_\infty^{-1/2},$$

then the gradient admits $\min_{t \in [T]} \mathbb{E} \left\| \nabla \mathcal{L}_2(z^t) \right\|_2 \leq \gamma$.

Furthermore, our result can be strengthened to the almost sure convergence by allowing for $\alpha = \alpha^t$ based on [16], [17].

III. Conclusion

This work is the first step towards the understanding of PIE and its variations. In the future, we plan to expand the analysis to the extended PIE for blind ptychography [6].

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