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The metaplectic action on modulation spaces

Hartmut Führ
Lehrstuhl A für Mathematik
RWTH Aachen University
fuehr@matha.rwth-aachen.de

Irina Shafkulovska
Faculty of Mathematics
University of Vienna
irina.shafkulovska@univie.ac.at

***Abstract*—We study the mapping properties of metaplectic operators $\widehat{S} \in \mathrm{Mp}(2d,\mathbb{R})$ on modulation spaces of the type $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$. Our main result is a full characterisation of the pairs $(\widehat{S}, \mathrm{M}^{p,q}(\mathbb{R}^d))$ for which the operator $\widehat{S} : \mathrm{M}^{p,q}(\mathbb{R}^d) \to \mathrm{M}^{p,q}(\mathbb{R}^d)$ is *(i)* well-defined, *(ii)* bounded. It turns out that these two properties are equivalent, and they entail that $\widehat{S}$ is a Banach space automorphism. Under mild conditions on the weight function, we provide a simple test to determine whether the well-definedness (boundedness) of $\widehat{S} : \mathrm{M}^{p,q}(\mathbb{R}^d) \to \mathrm{M}^{p,q}(\mathbb{R}^d)$ transfers to $\widehat{S} : \mathrm{M}^{p,q}_m(\mathbb{R}^d) \to \mathrm{M}^{p,q}_m(\mathbb{R}^d)$.**

I. Introduction

The aim of this note is to present results concerning the metaplectic invariance of modulation spaces. To our knowledge, this question was first raised in [1]. It is in fact a rather natural problem: On the one hand, the metaplectic group can be considered as the fundamental symmetry group of time-frequency analysis [2]–[4], and it is often employed to reduce the study of general cases to more specific, concrete settings. On the other hand, modulation spaces are among the chief objects of study in this area of mathematics, as witnessed by the recent book [5] and the numerous references therein.

The proofs of the following results will be presented in an upcoming paper; they were first obtained by the second author as part of her Master's thesis.

Basic definitions and notations

The cross-ambiguity function associated to $f, g \in L^2(\mathbb{R}^d)$ is the map

$$\begin{aligned}\mathrm{A}(f,g)(x,\omega) &= \int_{\mathbb{R}^d} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{-2\pi i\omega\cdot t}\,dt\\ &= \langle f, T_{x/2}M_\omega T_{x/2}g\rangle,\end{aligned}$$

where $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i\omega\cdot t}f(t)$ denote the time and the frequency shift, respectively. The latter equality allows to extend the definition of the ambiguity function for any $f \in \mathcal{S}'(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$.

In this paper, we consider modulation spaces associated to mixed-norm weighted L^p-spaces, more formally defined as the space $L^{p,q}_m(\mathbb{R}^{2d})$ of measurable functions $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ with the norm

$$\|f\|_{p,q,m} := \left\|\omega \mapsto \|f(\bullet,\omega)\, m(\bullet,\omega)\|_p\right\|_q,$$

where $m : \mathbb{R}^{2d} \to (0,\infty)$ is moderate, i.e., there is a $C > 0$ with

$$m(x+y) \le C\, v(x)\cdot m(y), \qquad x, y, \in \mathbb{R}^{2d}$$

for some submultiplicative weight v. This ensures the invariance of the space under shifts. We assume all weights in this paper to be locally bounded.

The modulation spaces associated to $L^{p,q}_m(\mathbb{R}^{2d})$ are then given by

$$\mathrm{M}^{p,q}_m(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid \mathrm{A}(f,g) \in L^{p,q}_m(\mathbb{R}^{2d}) \right\},$$

where $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is fixed. It is well-known that $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$ is independent of g, and that the associated norm

$$\|f\|_{\mathrm{M}^{p,q}_m(\mathbb{R}^d)} := \|\mathrm{A}(f,g)\|_{p,q,m}$$

is independent up to equivalence. Modulation spaces are Banach spaces, with natural, continuous embeddings as closed subspaces of $L^{p,q}_m(\mathbb{R}^{2d})$. The reconstruction formula is given by

$$\langle g,\gamma\rangle f = \mathrm{A}(\bullet, g)^*\, \mathrm{A}(f,\gamma),$$

where the adjoint acts as (in the weak sense)

$$\mathrm{A}(\bullet, g)^* F = \int_{\mathbb{R}^{2d}} F(x,\omega)\, T_{x/2}M_\omega T_{x/2}g\; d(x,\omega).$$

Both time and frequency shifts are representations of the abelian group $\mathbb{R}^d$, so they commute within their class, but symmetric time-frequency shifts $\rho(x,\omega) = T_{x/2}M_\omega T_{x/2} = M_{\omega/2}T_x M_{\omega/2}$ do not satisfy $\rho(\lambda)\rho(\nu) = \rho(\nu)\rho(\lambda)$ for arbitrary $\lambda, \nu \in \mathbb{R}^{2d}$. Instead, their commutator is given by

$$[\rho(\lambda), \rho(\nu)] = 1 - e^{\pi i\omega\cdot\eta - x\cdot\xi}, \quad \lambda = (x,\omega),\, \nu = (\eta,\xi).$$

This gives rise to the symplectic group $\mathrm{Sp}(2d,\mathbb{R}) \le \mathrm{SL}(2d,\mathbb{R})$ consisting of the matrices $S \in \mathrm{GL}(2d,\mathbb{R})$ satisfying

$$[\rho(S\lambda), \rho(S\nu)] = [\rho(\lambda)\rho(\nu)], \qquad \lambda, \nu \in \mathbb{R}^{2d}. \tag{1}$$

Equivalently, $S \in \mathrm{Sp}(2d,\mathbb{R})$ if and only if $S^T\mathcal{J}S = \mathcal{J}$, where $\mathcal{J}$ denotes the standard symplectic matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix},$$

and I_d is the identity matrix in $\mathbb{R}^{d\times d}$. The property (1) and methods from representation theory imply the existence of unitary operators $\widehat{S}$ with

$$\rho(S\lambda) = \widehat{S}\rho(\lambda)\widehat{S}^{-1}. \tag{2}$$

These operators generate the metaplectic group, denoted with $\mathrm{Mp}(2d,\mathbb{R})$. It is an explicit realisation of the double cover of $\mathrm{Sp}(2d,\mathbb{R})$ and the projection

$$\pi^{\mathrm{Mp}} : \mathrm{Mp}(2d,\mathbb{R}) \to \mathrm{Sp}(2d,\mathbb{R})$$

is a group homomorphism. By duality, the domain of a metaplectic operator can be extended to $\mathcal{S}'(\mathbb{R}^d)$. Standard examples of metaplectic operators are the dilations $D_L f(t) = |\det L|^{-1} f(L^{-1}t)$, the linear chirps $V_Q f(t) = e^{\pi i t\cdot Qt} f(t)$ and the Fourier transform. The deep connection between the ambiguity function and the metaplectic operators is best summarized by the symplectic covariance of the ambiguity function:

$$A(f,g)\circ S^{-1}\;(\lambda) = A(\widehat{S}f, \widehat{S}g)(\lambda). \tag{3}$$

II. Main Results

The central theorem concerns unweighted modulation spaces. The relation (3) prompts the introduction of the composition operator $\widetilde{S}F := F \circ S^{-1}$, for $F : \mathbb{R}^{2d} \to \mathbb{C}$. Given any Banach space Y of functions on $\mathbb{R}^{2d}$, the composition operator $\widetilde{S} : Y \to Y$ will be considered with its natural domain $D = \{F \in Y : F \circ S^{-1} \in Y\}$.

A. The characterization in the unweighted case

Theorem 2.1: Let $p, q \in [1, \infty]$, $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ be given. Let $S \in \mathrm{Sp}(2d, \mathbb{R})$ be the projection of $\widehat{S}$ onto $\mathrm{Sp}(2d, \mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R}^d)$ an arbitrary Schwartz function. Then the following statements are equivalent:

1) $\widehat{S} : \mathrm{M}^{p,q}(\mathbb{R}^d) \to \mathrm{M}^{p,q}(\mathbb{R}^d)$ is well-defined defined.
2) $\widehat{S} : \mathrm{M}^{p,q}(\mathbb{R}^d) \to \mathrm{M}^{p,q}(\mathbb{R}^d)$ is well-defined and bounded.
3) One of the following conditions holds:
    a) $p = q$, or
    b) $p \neq q$ and S is a block upper triangular matrix.

If one, hence all, of the statements hold, then $\widetilde{S} : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ is an isometric isomorphism (up to a multiplicative constant), and $\widehat{S}$ has a bounded inverse.

$$\begin{tikzcd}
\mathrm{M}^{p,q}(\mathbb{R}^d) \arrow[r, "\mathrm{A}(\cdot,g)"] \arrow[d, "\widehat{S}"'] & \mathrm{A}(\mathrm{M}^{p,q}(\mathbb{R}^d), g) \arrow[r, hook] \arrow[d, dashed] & L^{p,q}(\mathbb{R}^{2d}) \arrow[d, "\widetilde{S}"] \\
\mathrm{M}^{p,q}(\mathbb{R}^d) \arrow[r, "\mathrm{A}(\cdot,\widehat{S}^{-1}g)"'] & \mathrm{A}(\mathrm{M}^{p,q}(\mathbb{R}^d), \widehat{S}^{-1}g) \arrow[r, hook] & L^{p,q}(\mathbb{R}^{2d})
\end{tikzcd}$$

Fig. 1. Commutative diagram

Remark 2.2: Figure 1 summarizes the relationship of the various spaces and operators. We highlight several remarkable features of Theorem 2.1:

1) If the composition operator $\widetilde{S} : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ is everywhere defined, it is already a scalar multiple of an isometry.
2) The commutative diagram in Figure 1 allows to conclude that boundedness of the composition operator $\widetilde{S} : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ already implies boundedness of $\widehat{S}$. Theorem 2.1 states an important and highly nontrivial converse, implying that the boundedness properties of the composition operator $\widetilde{S} : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ are completely determined by its restriction to the proper closed subspace $\mathrm{A}(\mathrm{M}^{p,q}(\mathbb{R}^d), g)$.

B. Implications for the weighted case

We now turn to the metaplectic action on *weighted* modulation spaces. The following results concentrate on the transfer between the weighted and unweighted cases. A concrete characterization of bounded metaplectic operators $\widehat{S}$ on $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$ in terms of the matrix S is obtainable by combining Theorem 2.4 with 2.1.

We first consider the composition operators $\widetilde{S}_m : L^{p,q}_m(\mathbb{R}^{2d}) \to L^{p,q}_m(\mathbb{R}^{2d})$ for a suitable weight m; the subscript is needed to mark a contrast to $\widetilde{S} : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$. Note that there exists a simple isometric isomorphism between weighted an unweighted spaces,

$$\Phi_m : L^{p,q}_m(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d}), \quad f \mapsto f \cdot m \ ,$$

which allows to analyze $\widetilde{S}_m$ in terms of $\widetilde{S}$. This gives rise to the following result.

Theorem 2.3: Let v be submultiplicative and m a v-moderate weight function. Let $S \in \mathrm{Sp}(2d, \mathbb{R})$ be given. Then the following statements hold. Note that the assumptions in items 1.) and 2.) really formulate a dichotomy.

1) If $\widetilde{S}$ is bounded, then $\widetilde{S}_m$ is bounded if and only if
$$C_m := \operatorname*{ess\,sup}_{z \in \mathbb{R}^{2d}} \frac{m}{m \circ S^{-1}} < \infty.$$
2) If $\widetilde{S}$ is not everywhere defined, and
$$T_m := \operatorname*{ess\,inf}_{z \in \mathbb{R}^{2d}} \frac{m(z)}{m \circ S^{-1}(z)} > 0,$$
then $\widetilde{S}_m$ is not everywhere defined.

Before we finally address the modulation space setting, we need one more piece of terminology. Two weights m_1 and m_2 are called *equivalent* (denoted with $m_1 \asymp m_2$) if there exist positive constants $0 < c \leq C < \infty$ satisfying

$$c\,m_1 \leq m_2 \leq C\,m_1.$$

Theorem 2.4: Let $v(z) = (1 + \|z\|)^N$ for some $N \in \mathbb{N}$ and m be a v-moderate, even weight, $p, q \in [1, \infty]$. If $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ with projection $\pi^{Mp}(\widehat{S}) = S$ satisfies $m \asymp m \circ S^{-1}$, then the following statements are equivalent:

1) $\widehat{S}$ is a bounded operator from $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$ to $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$.
2) $\widehat{S}$ is a bounded operator from $\mathrm{M}^{p,q}(\mathbb{R}^d)$ to $\mathrm{M}^{p,q}(\mathbb{R}^d)$.
3) One of the following conditions holds:
    a) $p = q$, or
    b) $p \neq q$ and S is a block upper triangular matrix.

If one, hence all, of the statements hold, then $\widehat{S}$ is a Banach space automorphism on $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$.
For the proof we use 2.1, 2.3 and a lift from $\mathrm{M}^{p,q}_m(\mathbb{R}^d)$ onto $\mathrm{M}^{p,q}(\mathbb{R}^d)$. The (quite nontrivial) lifting step is based on techniques and results of [6].

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