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Numerical Methods for the Regularized Beckmann Problem

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***Abstract*—We analyze optimal transport problems in the so-called Beckmann form, where we seek a transport flow with minimal cost between two marginals that are probability measures on compact subsets of Euclidean space. Similarly to the Benamou-Brenier formulation of optimal transport, the Beckmann formulation allows to construct time dependent interpolations between the marginals with the upside that in the discrete setting the timesteps don't need to be fixed a priori. To ensure uniqueness of the solution, we employ L^α-regularization. We focus on the numerical treatment of the regularized Beckmann problem with non-uniform cost.**

I. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a compact Lipschitz domain and let $\mu^+, \mu^- \in \mathfrak{M}(\Omega)$ be two Radon measures on Ω that satisfy

$$\mu^+(\Omega) = \mu^-(\Omega)\,.$$

The so-called Beckmann problem seeks a vector valued measure $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ satisfying $\operatorname{div} q = \mu^+ - \mu^- =: \mu$ (in a suitable weak sense) such that the objective

$$\int_\Omega w(x)\,\mathrm{d}|q|(x) \tag{1}$$

is minimized for some continuous function $w \geq 0$. This can be understood as minimizing the cost of transporting the mass μ^+ onto the mass μ^-, where transporting one unit of mass through a point $x \in \Omega$ hast a cost of $w(x)$.

The Beckmann problem is closely related to other problems of optimal transport theory, namely the so called Monge problem and the Kantorovich problem as well as the Monge-Kantorovich equation [1], [2].

Existence of solutions for the Beckmann problem is well known [1], [3], but since the objective functional (1) is not strictly convex, solutions may not be unique. Moreover, for general $\mu^+, \mu^- \in \mathfrak{M}(\Omega)$, a solution may not admit a density.

We therefore add a strictly convex regularization term that forces the solution to be both unique and a function in $\mathrm{L}^\alpha(\Omega)$. The problem now reads as

$$\min_{\substack{q \in \mathrm{L}^\alpha(\Omega)\,,\\ \operatorname{div} q = \mu}} \int_\Omega w(x)|q(x)|\,\mathrm{d}x + \frac{\varepsilon}{\alpha}\|q\|^\alpha_{\mathrm{L}^\alpha(\Omega,\mathbb{R}^d)}\,, \tag{BP}$$

where the divergence operator has to be defined in a suitable way and $\varepsilon > 0$ is a regularization parameter.

II. Semi-Smooth Newton

Besides enforcing uniqueness and regularity, the regularization term also provides an entry point for a numerical approach for solving problem (BP). We observe that the first order necessary optimality conditions for (BP) are given by

$$\begin{aligned} \varepsilon|q|^{\alpha-2}q + \partial\|q\|_{1,w} + \operatorname{grad} y &= 0 \\ \operatorname{div} q &= \mu\,, \end{aligned} \tag{2}$$

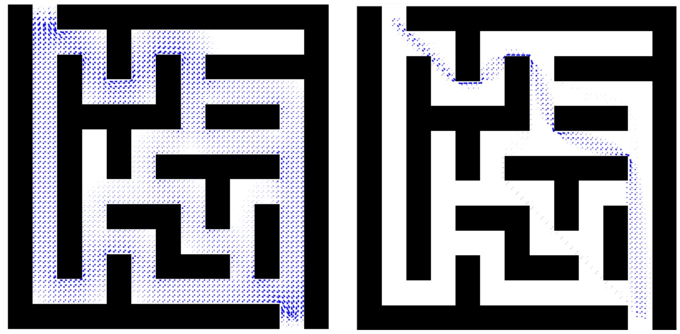

Fig. 1. Numerical example for the semi-smooth Newton method. Black areas denote a high transportation cost w, white areas a low cost. The blue arrows represent the flow q. Left: $\varepsilon = 0.005$. Right: $\varepsilon = 0.5$.

where $\|q(x)\|_{1,w} := w(x)|q(x)|$ and y is a Lagrange multiplier in the Sobolev space $\mathrm{W}^{1,\alpha'}(\Omega)$. Thanks to the regularization, the term

$$\varepsilon|q|^{\alpha-2}q + \partial\|q\|_{1,w}$$

is invertible, so that we can eliminate q from (2) and obtain

$$\operatorname{div} F(-\operatorname{grad} y) = \mu\,, \tag{3}$$

with

$$F(x,q) = \frac{q}{|q|}\left(\frac{1}{\varepsilon}(|q| - w)_+\right)^{\alpha'-1}.$$

Of course, (3) has to be understood in a suitable week sense. The idea is now that if F is regular enough, we can derive a semi-smooth Newton iteration from (3) which, due to the weak interpretation, is well suited to be solved in the discrete setting via standard finite element methods. Because $\operatorname{grad}_q F(x,q)$ vanishes for $|q| \leq w(x)$ due to the positive part, we employ a second regularization term of Huber type.

A numerical example with varying regularization parameter ε is shown in Fig. 1.

III. Approximation Results

In addition to a numerical scheme, we also obtain convergence results for vanishing regularization parameters. This guarantees that the numerical scheme gives good approximations of the unregularized problem for small enough regularization parameters.

References

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