On connections between Amplitude Flow and Error Reduction for phase retrieval

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***Abstract*—In this paper, we consider two iterative algorithms for the phase retrieval problem: the well-known Error Reduction method and the Amplitude Flow algorithm, which performs minimization of the amplitude-based squared loss via the gradient descent. We show that Error Reduction can be interpreted as a quasi-Newton method applied to minimize the same amplitude-based squared loss, which allows to establish its convergence properties. Moreover, we show that for a class of measurement scenarios two methods have the same computational complexity and sometimes even coincide.**

I. Introduction

The problem of phaseless recovery considers the reconstruction of an unknown $x \in \mathbb{C}^d$ from a quadratic measurements of the form

$$y_j = |(Ax)_j|^2 + n_j, \quad j \in [m], \tag{1}$$

with $A \in \mathbb{C}^{m \times d}, m > d$ denoting the measurement matrix, n – the noise and $[m] := \{1, \dots, m\}$. It has many application in the fields of crystallography [1], noncrystalline materials [2], [3], [4] and optical imaging [5].

One of the popular approaches to the recovery of x is the Error Reduction (ER) method [6], [7], [8]. It poses the phase retrieval problem as a problem of finding Ax, which belongs to the intersection of two sets

$$\mathcal{M} := \{u \in \mathbb{C}^m : |u_j|^2 = y_j,\ j \in [m]\},$$
$$\operatorname{im}(A) := \{u \in \mathbb{C}^m : u = Az \text{ for some } z \in \mathbb{C}^d\}.$$

Starting with a guess $z^0 \in \mathbb{C}^d$, ER constructs the initial iterate $u^0 = Az^0$ and then proceeds by consequently projecting u^t onto $\mathcal{M}$ and $\operatorname{im}(A)$, so that $u^{t+1} = P_{\operatorname{im}(A)} P_{\mathcal{M}} u^t, t \geq 0$, where P denotes the projection operator. Note, that set $\mathcal{M}$ is non-convex and, thus, convergence of the algorithm is not guaranteed theoretically. The iterations are continued until the algorithm finds a fixed point $u^{t+1} = u^t$ and an estimate z^t corresponding to the least squares solution $u^t = Az^t$ is returned. When iterations of ER are rewritten with respect to z^t, it grants the following updates

$$z^{t+1} = A^\dagger \operatorname{diag}\left(\frac{\sqrt{y}}{|Az^t|}\right) Az^t, \quad t \geq 0, \tag{ER}$$

where $\operatorname{diag}(u)$ denotes diagonal matrix with vector u on main diagonal, $A^\dagger$ is the Moore-Penrose inverse, and the division is performed entrywise. We note that the case $(Az^t)_j = 0$ for some $j \in [m]$ is special, since there are multiple possible projections on set $\mathcal{M}$. In such a scenario, 0 is commonly mapped to $e^{i\varphi}\sqrt{y_j}$ with $\varphi \in [0, 2\pi)$ either some fixed or randomly chosen value. In this paper we map 0 to 0 – an average value of all possible projections, which corresponds to setting the fraction $(Az^t)_j / |(Az^t)_j|$ as zero.

Another popular algorithm is Amplitude Flow (AF) [9], [10], [11], [12], which solves the phase retrieval problem by minimizing the amplitude-based squared loss

$$L_2(z) := \sum_{j=1}^m (|Az| - \sqrt{y_j})^2$$

via gradient descent. The generalized gradient of L_2 is given by

$$\nabla L_2(z) = A^* \operatorname{diag}\left(1 - \frac{\sqrt{y}}{|Az|}\right) Az,$$

with A^* denoting conjugate transpose of A. Then, starting with an initial guess z^0, the iterates of AF are generated by

$$z^{t+1} = z^t - \mu \nabla L_2(z^t), \quad t \geq 0. \tag{AF}$$

and algorithm terminates, once a fixed (critical) point $z^{t+1} = z^t$ is reached. When the learning rate μ is chosen to be the squared inverse of the spectral norm $\|A\|^{-2}$, AF has a guaranteed convergence rate of $\mathcal{O}(t^{-1/2})$.

II. Main Results

Our two main results show that the two algorithms ER and AF are related and have the same sets of fixed points provided the measurement matrices are non-singular.

Theorem II.1. *Let* $\operatorname{rank}(A) = d$*. Then, ER has convergence rate of* $\mathcal{O}(t^{-1/2})$*.*

Theorem II.2. *Let* $\operatorname{rank}(A) = d$*. Then,* $z \in \mathbb{C}^d$ *is a fixed point of ER if and only if* z *is the fixed point of AF.*

At the core of our results is the following lemma.

Lemma II.3. *Let* $\operatorname{rank}(A) = d$*. Then, iterations of ER are quasi-Newton iterations given by*

$$z^{t+1} = z^t - (A^*A)^{-1} \nabla L_2(z^t), \quad t \geq 0.$$

From these results it may seem that two algorithms are comparable. However, a single iteration of ER requires computing the pseudoinverse, which is considerably slower than AF. The next corollary shows, that this difference is less significant when the columns of A are orthogonal and, in fact, the two algorithms are equivalent.

Corollary II.4. *Let* $A^*A = \operatorname{diag}(v)$ *for some* $v \in \mathbb{R}^d$*,* $v_j > 0, j \in [d]$*. Then, ER has the same computational complexity as AF. Furthermore, if* $v_j = c$*,* $c > 0$ *for all* $j \in [m]$*, two algorithms coincide.*

An important example where the conditions of Corollary II.4 are satisfied is ptychography [13], [14], [15], [16], where A represents a discrete Short-time Fourier transform.

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