Wavelets on Intervals Derived from Arbitrary Compactly Supported Biorthogonal Multiwavelets

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***Abstract*—The construction of multiwavelets on a bounded interval, which can preserve most of their original properties on $\mathbb{R}$, is fundamental in theory and applications (e.g., images and numerical PDEs). Except for a few concrete examples, no systematic construction method for multiwavelets on a bounded interval is available. Hence, given any compactly supported biorthogonal multiwavelet on $\mathbb{R}$, we present a direct approach, to construct all possible locally supported biorthogonal multiwavelets on [0,1] satisfying prescribed vanishing moments, polynomial reproduction, and homogeneous boundary conditions. It neither explicitly involves dual refinable functions nor dual multiwavelets. For the sake of illustration, some examples of orthogonal and biorthogonal multiwavelets constructed on [0,1] will be given.**

I. Introduction

There are many interesting studies in the literature on the construction of wavelets on bounded intervals (see references in [1]). This paper focuses on the direct approach proposed in [1] to construct all possible locally supported biorthogonal multiwavelets on $[0,1]$. Not having to explicitly deal with dual refinable functions and multiwavelets makes it more convenient to use than the classical approach also discussed in [1]. We restate several definitions and notations. For $J \in \mathbb{Z}$, the multiwavelet affine system $\mathsf{AS}_J(\phi;\psi)$, generated by ϕ and ψ through dyadic dilation and integer shifts, is defined to be

$$\begin{aligned}\mathsf{AS}_J(\phi;\psi) := &\{\phi^\ell_{J;k} \ : \ k \in \mathbb{Z}, \ell = 1,\dots,r\}\\ &\cup \{\psi^\ell_{j;k} \ : \ j \geqslant J, k \in \mathbb{Z}, \ell = 1,\dots,s\},\end{aligned}$$

where $\phi^\ell_{J;k} := 2^{J/2}\phi^\ell(2^J\cdot - k)$ and $\psi^\ell_{j;k} := 2^{j/2}\psi^\ell(2^j\cdot - k)$. We say that $\{\phi;\psi\}$ is *a Riesz multiwavelet in* $L_2(\mathbb{R})$ if $\mathsf{AS}_0(\phi;\psi)$ is a Riesz basis of $L_2(\mathbb{R})$. We restate that $(\{\tilde\phi;\tilde\psi\},\{\phi;\psi\})$ is *a biorthogonal multiwavelet in* $L_2(\mathbb{R})$ if both $\{\tilde\phi;\tilde\psi\}$ and $\{\phi;\psi\}$ are Riesz multiwavelets in $L_2(\mathbb{R})$ such that $\mathsf{AS}_0(\tilde\phi;\tilde\psi)$ and $\mathsf{AS}_0(\phi;\psi)$ are biorthogonal to each other in $L_2(\mathbb{R})$. For a compactly supported vector function ϕ (or a finitely supported filter $a \in (l_0(\mathbb{Z}))^{r\times s}$), we define $\mathrm{fsupp}(\phi)$ (or $\mathrm{fsupp}(a)$) to be the shortest interval with *integer endpoints* such that ϕ (or a) vanishes outside $\mathrm{fsupp}(\phi)$ (or $\mathrm{fsupp}(a)$). For a compactly supported vector function ψ, we say that ψ has m *vanishing moments* if $\int_{\mathbb{R}} x^j\psi(x)dx = 0$ for all $j = 0,\dots,m-1$. In particular, we define $\mathrm{vm}(\psi) := m$ with m being the largest such integer. Lastly, we define $\mathsf{S}_j(H) := \overline{\mathrm{span}\{f(2^j\cdot) \ : \ f \in H\}}$ for $j \in \mathbb{Z}, H \subseteq L_2(\mathbb{R})$, where the overhead bar refers to closure in $L_2(\mathbb{R})$.

II. Main Results

In this section, we restate a key theorem for our proposed direct approach. For more details, see [1, Section 4].

Theorem II.1. *[1, Theorem 4.2] Let $(\{\tilde\phi;\tilde\psi\},\{\phi;\psi\})$ be a compactly supported biorthogonal wavelet in $L_2(\mathbb{R})$ with a biorthogonal wavelet filter bank $(\{\tilde a;\tilde b\},\{a;b\})$ satisfying items (1)–(4) of [1, Theorem 2.1]. Suppose that $\Phi = \{\phi^L\} \cup \{\phi(\cdot - k) \ : \ k \geqslant n_\phi\} \subseteq L_2([0,\infty))$ satisfies item (i) of [1, Theorem 2.7], Φ is a Riesz sequence in $L_2([0,\infty))$, and ϕ^L is a compactly supported vector function in $L_2([0,\infty)) \cap H^\tau(\mathbb{R})$ for some $\tau > 0$. Take an integer $n_\psi \geqslant \max(-l_\psi, \frac{n_\phi - l_b}{2})$ and define $m_\phi := \max(2n_\phi + h_{\tilde a}, 2n_\psi + h_{\tilde b})$, where $[l_{\tilde a}, h_{\tilde a}] := \mathrm{fsupp}(\tilde a)$ and $[l_{\tilde b}, h_{\tilde b}] := \mathrm{fsupp}(\tilde b)$. Let $m, n_0 \in \mathbb{N}_0$. Construct $\Psi := \{\psi^L\} \cup \{\psi(\cdot - k) \ : \ k \geqslant n_\psi\}$ such that*

(i) *$\psi^L \subseteq \mathrm{span}(\{\phi^L(2\cdot)\} \cup \{\phi(2\cdot - k) \ : \ n_\phi \leqslant k < m_\phi + n_0\})$, ψ^L has m vanishing moments with $\mathrm{vm}(\psi^L) \geqslant m$, and the set ψ^L (which is regarded as in $\mathsf{S}_1(\Phi)/\mathsf{S}_0(\Phi \cup \{\psi(\cdot - k) \ : \ k \geqslant n_\psi\})$) is a basis of the finite-dimensional quotient space $\mathsf{S}_1(\Phi)/\mathsf{S}_0(\Phi \cup \{\psi(\cdot - k) \ : \ k \geqslant n_\psi\})$.*

(ii) *Every element in $\phi^L(2\cdot) \cup \phi^E(2\cdot)$ is a finite linear combination of elements in $\Phi \cup \Psi$, where $\phi^E := \{\phi(\cdot - k) \ : \ n_\phi \leqslant k < m_\phi\}$. That is, for some integers $h_C \geqslant n_\phi$ and $h_D \geqslant n_\psi$,*

$$\begin{aligned}\begin{bmatrix}\phi^L(2\cdot)\\ \phi^E(2\cdot)\end{bmatrix} = A_0\phi^L + B_0\psi^L + \sum_{n_\phi \leqslant k < h_C} C(k)\phi(\cdot - k)\\ + \sum_{n_\psi \leqslant k < h_D} D(k)\psi(\cdot - k)\end{aligned} \tag{1}$$

for some matrices A_0, B_0, $C(k)$ with $n_\phi \leqslant k < h_C$, and $D(k)$ with $n_\psi \leqslant k < h_D$.

Define $n_{\tilde\phi} := \max(m_\phi, h_C, -l_{\tilde\phi}, 1 - l_{\tilde a})$ and $n_{\tilde\psi} := \max(n_\psi, h_D, -l_{\tilde\psi}, \lceil\frac{n_{\tilde\phi} - l_{\tilde b} + 1}{2}\rceil)$. Then we must have

$$\phi(2\cdot - k_0) = \sum_{k=n_{\tilde\phi}}^{n_{\tilde\phi}-1} \overline{\tilde a(k_0 - 2k)}^{\mathsf{T}}\phi(\cdot - k) + \sum_{k=n_{\tilde\psi}}^{n_{\tilde\psi}-1} \overline{\tilde b(k_0 - 2k)}^{\mathsf{T}}\psi(\cdot - k). \tag{2}$$

for all $m_\phi \leqslant k_0 < n_{\tilde\phi}$. Now we can rewrite/combine (1) and (2) together into the following equivalent form: $\mathring\phi^L(2\cdot) = \overline{\tilde A_L}^{\mathsf{T}}\mathring\phi^L + \overline{\tilde B_L}^{\mathsf{T}}\mathring\psi^L$, where $\mathring\phi^L := \{\phi^L\} \cup \{\phi(\cdot - k) \ : \ n_\phi \leqslant k < n_{\tilde\phi}\}$, $\mathring\psi^L := \{\psi^L\} \cup \{\psi(\cdot - k) \ : \ n_\psi \leqslant k < n_{\tilde\psi}\}$, and the matrices $\tilde A_L, \tilde B_L$ are uniquely determined by $A_0, B_0, \{C(k)\}_{k=n_\phi}^{h_C-1}, \{D(k)\}_{k=n_\psi}^{h_D-1}$ and the filters $\tilde a, \tilde b$. If the spectral radius of $\tilde A_L$ is less than $2^{-1/2}$ then the following $\tilde\phi^L$ and $\tilde\psi^L$ are well-defined compactly supported vector functions in $L_2([0,\infty))$:

$$\tilde\phi^L := \sum_{j=1}^{\infty} 2^{j-1}\tilde A_L^{j-1}\tilde g(2^j\cdot) \quad \text{with} \quad \tilde g := 2\sum_{k=n_{\tilde\phi}}^{\infty}\tilde A(k)\tilde\phi(\cdot - k),$$

$$\tilde\psi^L := 2\tilde B_L\tilde\phi^L(2\cdot) + 2\sum_{k=n_{\tilde\phi}}^{\infty}\tilde B(k)\tilde\phi(2\cdot - k).$$

Then $\mathsf{AS}_J(\tilde\Phi;\tilde\Psi)_{[0,\infty)}$ and $\mathsf{AS}_J(\Phi;\Psi)_{[0,\infty)}$ form a pair of biorthogonal Riesz bases of $L_2([0,\infty))$ for every $J \in \mathbb{Z}$, where $\tilde\Phi := \{\tilde\phi^L\} \cup \{\tilde\phi(\cdot - k) \ : \ k \geqslant n_{\tilde\phi}\}$ and $\tilde\Psi := \{\tilde\psi^L\} \cup \{\tilde\psi(\cdot - k) \ : \ k \geqslant n_{\tilde\psi}\}$.

REFERENCES

[1] B. Han and M. Michelle, Wavelets on Intervals Derived from Arbitrary Compactly Supported Biorthogonal Multiwavelets. *Appl. Comput. Harmon. Anal.* **53** (2021), 270–331.