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Classification of Lower Frame and Riesz-Fischer Sequences by Unbounded Operators

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***Abstract*—In this paper, we and investigate the properties of (possibly unbounded) frame-related operators of an arbitrary sequence. In particular we characterize all Riesz-Fischer sequences and lower frame sequences by (a particular version of) the synthesis operators.**

I. Introduction

Unbounded operators are an interesting and important topic in operator theory. They have many applications in physics, especially in quantum mechanics. Clearly unbounded operators introduce more technicalities. e.g. it is known since 1976 that a matrix representation of unbounded operator using orthonormal bases can result in problems.

In this paper we will link (generalized) frame theory with the theory of unbounded operators. While for frames, the frame-related operators (synthesis, analysis, frame and Gram operators) are always bounded, for other sequences they can always be defined as unbounded operators. And so the non-frame sequences are more naturally related to unbounded operators.

The analysis operator of a sequence is always closed and so for this the related theory is still rather 'tame'. This is not the case for the synthesis operator. We tackled those harder technicalities and classify Riesz-Fischer sequences and lower frame sequences by properties of the synthesis operator alone.

Although there are some characterization for non-frame sequence Ψ by $D_\Psi^{-1}, D_\Psi^*, C_\Psi, S_\Psi$ and G_Ψ [1], some questions remain open for example *Can we classify Riesz-Fisher sequences and lower frame sequences by properties of D_Ψ alone, in particular not including D_Ψ^{-1} or D_Ψ^*?*

II. Main Results

Assume that $\mathcal{H}$ is a separable Hilbert space. For a sequence Ψ, we can define the synthesis related operator $D_\Psi : \mathsf{dom}\,(D_\Psi) \subseteq \ell^2 \to \mathcal{H}$, where $\mathsf{dom}\,(D_\Psi) = \{(c_i)_i \in \ell^2; \sum_{i\in I} c_i\psi_i \text{ converges}\}$. Similarly, C_Ψ can be defined on its domain.

By using the potentially unbounded operators C_Ψ and D_Ψ we have the following definition.

Definition 2.1: Let $\Psi = \{\psi_i\}_{i\in I} \subseteq \mathcal{H}$. We say that

1) Ψ is a *lower frame sequence*, if and only if there exists a constant $A > 0$ such that $\sqrt{A}\cdot\|f\| \leq \|C_\Psi f\|$ for all $f \in \mathsf{dom}\,(C_\Psi)$.
2) Ψ is an *exact* sequence if for every $k \in I$ we have that
$$\overline{span}_{i\in I}\{\psi_i\} \neq \overline{span}_{i\in I, i\neq k}\{\psi_i\}.$$
In particular, a lower frame sequence is exact if it stops to be a lower frame sequence when an arbitrary element is removed.
3) Ψ is a *Riesz-Fischer sequence*, if and only if there exists a constant $A > 0$ such that $\sqrt{A}\cdot\|c\| \leq \|D_\Psi c\|$ for all finite scalar sequences $c = \{c_i\}_{i\in I} \in \mathsf{dom}\,(D_\Psi)$.

It is worthwhile to mention that if Ψ is a sequence in $\mathcal{H}$ then by the restriction of the Hilbert space $\mathcal{H}$ to $\mathcal{H}_\Psi = \overline{\mathsf{dom}\,(C_\Psi)}$ with the topology induced by $\mathcal{H}$, the operator $C_\Psi : \mathsf{dom}\,(C_\Psi) \subseteq \mathcal{H}_\Psi \to \ell^2$ is a densely defined operator and the associated synthesis operator $\mathbf{D}_\Psi : \mathsf{dom}\left(D_{\pi_{\mathcal{H}_\Psi}\Psi}\right) \subseteq \ell^2 \to \mathcal{H}_\Psi$ is always closable. One can see that $\pi_{\mathcal{H}_\Psi} D_\Psi = \mathbf{D}_\Psi \subset C_\Psi^\times$. Here we denote by $\cdot^\times$ the adjoint with regards to $\mathcal{H}_\Psi$. If D_Ψ is a closable operator, then $\mathbf{D}_\Psi$ and D_Ψ are the same.

In the following result, we can characterize lower frame and Riesz-Fischer sequences without assume their synthesis operator is closable (as done in [3]).

Theorem 2.2: Assume that Ψ is a lower frame sequence in $\mathcal{H}$. The following assertions hold.

1) Ψ is a lower frame sequence in $\mathcal{H}$ if and only if $\pi_{\mathcal{H}_\Psi}\Psi$ is a lower frame sequence for $\mathcal{H}$ if and only if $\overline{\mathbf{D}}_\Psi$ is surjective.
2) Ψ is a Riesz-Fischer sequence in $\mathcal{H}$ if and only if $\pi_{\mathcal{H}_\Psi}\Psi$ is a Riesz-Fischer sequence for $\mathcal{H}$ if and only if $\overline{\mathbf{D}}_\Psi$ is injective and has closed range.

Here we present some equivalent conditions for a lower frame sequence to be a complete Riesz-Fischer sequence. The similar case for frames can be found at [2, Theorem 6.1.1].

Theorem 2.3: Let Ψ be a lower frame sequence with closable synthesis operator. If $\overline{\mathbf{D}}_\Psi$ has closed range, then the following are equivalent:

1) Ψ is a complete Riesz-Fischer sequence.
2) Ψ is minimal.
3) Ψ is ω-independent.
4) Ψ has a biorthogonal sequence.
5) Ψ has a unique biorthogonal sequence.
6) Ψ is an exact sequence.
7) Ψ is a basis in the weak sense.

III. Conclusion

We have a full characterization for lower frame sequences and Riesz-Fischer sequences by the synthesis operators of sequences. Proofs of those statements can be found in an upcoming paper. There still are many open questions regarding to reconstruction formula or equivalently duality concept that seems as an interesting challenge.

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References

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