Mean-dispersion principles and the Wigner transform

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We prove an uncertainty principle for an orthonormal sequence $\{f_k\}_{k\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$ related to the Wigner transform

$$W(f_k)(x,\xi) = \frac{1}{\sqrt{2\pi}} \int f_k\left(x + \frac{t}{2}\right) \overline{f_k\left(x - \frac{t}{2}\right)} e^{-it\xi} dt.$$

More precisely, given the following operator (dual, by means of Fourier transform, to the twisted laplacian operator)

$$\hat{L} := \left(\frac{1}{2}D_\xi + x\right)^2 + \left(\frac{1}{2}D_x - \xi\right)^2,$$

we have that

$$\sum_{k=0}^{n} \langle \hat{L} W(f_k), W(f_k)\rangle \geq (n+1)^2, \qquad \forall n \in \mathbb{N}_0.$$

Moreover equality holds for $0 \leq n \leq n_0$, if and only if all f_k with $0 \leq k \leq n_0$ are equal, up to a unitary constant, to the Hermite functions h_k defined by

$$h_k(t) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-t^2/2} H_k(t), \quad H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2}, \quad k \in \mathbb{N}_0.$$

The above uncertainty principle implies, as a trivial consequence, *Shapiro's mean-dispersion principle*: there does not exist an infinite orthonormal sequence $\{f_k\}_{k\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$ such that all means $\mu(f_k)$, $\mu(\hat{f}_k)$ and variances $\Delta(f_k)$, $\Delta(\hat{f}_k)$ associated to f_k and their Fourier transform $\hat{f}_k$ are uniformly bounded. Here

$$\mu(f) := \frac{1}{\|f\|^2} \int t|f(t)|^2 dt, \quad \Delta(f) := \frac{1}{\|f\|} \left(\int |t - \mu(f)|^2 |f(t)|^2 dt \right)^{1/2}, \quad f \in L^2(\mathbb{R}).$$