Obstructions for Gabor frames of the second order B-spline

Riya Ghosh
Indian Institute of Technology (ISM) Dhanbad, India.

Abstract

One of the fundamental problems in Gabor analysis is to determine the values of $a, b > 0$ such that the Gabor system $\mathcal{G}(g, a, b) = \{e^{2\pi i b m \cdot} g(\cdot - ak) : k, m \in \mathbb{Z}\}$ for a nonzero window function $g \in L^2(\mathbb{R})$ forms a frame for $L^2(\mathbb{R})$. For the window function g, the set $\mathcal{F}(g) = \big\{(a, b) \in \mathbb{R}^2_+ : \mathcal{G}(g, a, b) \text{ is a frame}\big\}$ is called the frame set of g. Determining the Gabor frame set for a given window is a challenging open problem in time-frequency analysis. In particular, the frame set for B-splines has many obstructions. Especially, the second order B-spline $Q_2 = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) * \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$ is the most important object in the study of Gabor analysis. Lemvig and Nielsen (2016) conjectured that if

$$a_0 = \frac{1}{2m+1},\ b_0 = \frac{2k+1}{2},\ k, m \in \mathbb{N},\ k > m,\ a_0 b_0 < 1,$$

then the Gabor system $\mathcal{G}(Q_2, a, b)$ of the second order B-spline Q_2 is not a frame along the hyperbolas

$$ab = \frac{2k+1}{2(2m+1)},\ \text{for } b \in \left[b_0 - a_0 \frac{k-m}{2}, b_0 + a_0 \frac{k-m}{2}\right],$$

for every a_0, b_0. Nielsen (2015) also conjectured that $\mathcal{G}(Q_2, a, b)$ is not a frame for

$$a = \frac{1}{2m},\ b = \frac{2k+1}{2},\ k, m \in \mathbb{N},\ k > m,\ ab < 1 \text{ with } \gcd(4m, 2k+1) = 1.$$

Using the Laurent operator technique, we prove that both conjectures are true. This study makes significant progress in characterizing the frame set of a second-order B-spline with rational density. This is a joint work with A Antony Selvan.

[1] Lemvig, J. and Nielsen, K.H. *Counterexamples to the B-spline conjecture for Gabor frames*, J. Fourier Anal. Appl., 22, 1440-1451, 2016.

[2] Nielsen, K.H. *The frame set of Gabor systems with B-spline generators*, 2015, https://www2.imm.dtu.dk/pubdb/edoc/imm6922.pdf.