Weaving properties of compact-tight and approximated frames

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Abstract

In this note, we study weaving properties of both compact tight-frames and approximated frames. Compact tight frames are defined as frames whose frame operator is a perturbation of constant multiple of the identity. More precisely, the frame operator has the form $S = cI + K$ with K being a compact operator. We also give some results from ϵ-approximation in the setting of weaving frames.

1 Introduction

A sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space H is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

The numbers A and B are called *lower* and *upper frame bounds*, respectively. If we can choose $A = B$, the frame is called *tight*, and if $A = B = 1$, the frame is called *Parseval*. The sequence $\{f_k\}_{k=1}^{\infty}$ is said to be a *Bessel sequence* if at least the upper frame condition holds.

Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ be a frame for H. Then, the operators $T_{\mathcal{F}}$ and $T_{\mathcal{F}}^*$ defined by

$$T_{\mathcal{F}} : \ell^2 \to H, \quad T_{\mathcal{F}}(\{c_k\}_{k\in I}) = \sum_{k=1}^{\infty} c_k f_k,$$

and

$$T_{\mathcal{F}}^* : H \to \ell^2, \quad T_{\mathcal{F}}^* f = \{\langle f, f_k\rangle\}_{k\in I}$$

are known as the *synthesis* and *analysis* operators of $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$, respectively. The *frame operator* $S : H \to H$, defined by

$$Sf = T_{\mathcal{F}} T_{\mathcal{F}}^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k,$$

is a positive, self-adjoint and invertible operator. For more information about frames we refer the reader to [2].

Weaving frames [1] have fundamental role in this paper.

Definition 1.1. [1] A finite family of frames $\{f_{ij}\}_{j=1,i\in I}^{M}$ in a Hilbert space H is said to be woven if there exist universal constants A and B such that for every partition $\{\sigma_j\}_{j=1}^{M}$ of I, the family $\{f_{ij}\}_{j=1,i\in\sigma_j}^{M}$ is a frame for H with bounds A and B, respectively. Each family $\{f_{ij}\}_{j=1,i\in\sigma_j}^{M}$ is called a weaving.

Definition 1.2. [3] Let $\{f_k\}_{k\in I}$ be a frame for H. Given any $\epsilon > 0$, a sequence $\{\tilde{f}_k\}_{k\in I} \subset H$ is called an ϵ-approximation of $\{f_k\}_{k\in I}$ if

$$\|\sum c_k (f_k - \tilde{f}_k)\|^2 \le \epsilon \sum |c_k|^2$$

for all finite sequences $\{c_k\}$.

2 Woven ϵ-approximations of frames

In the following we show that a frame can be woven with its ϵ-approximation.

Theorem 2.1. *Let $\{f_k\}_{k\in I}$ be a frame for H with bounds A, B and ϵ-approximation $\tilde{\mathcal{F}} = \{\tilde{f}_k\}_{k\in I}$ such that $\epsilon < \sqrt{A}$. Then $\{f_k\}_{k\in I}$ is woven with $\{\tilde{f}_k\}_{k\in I}$.*

Proof. Let $T_{\mathcal{F}J^c}$ and $T_{\tilde{\mathcal{F}}J^c}$ be the synthesis operators of Bessel sequences $\{f_k\}_{k\in\sigma^c}$ and $\{\tilde{f}_k\}_{k\in\sigma^c}$, respectively. Then for every $\sigma \subseteq I$ and $f \in H$, we have

$$\begin{aligned}
\left(\sum_{k\in\sigma} |\langle f, f_k\rangle|^2 + \sum_{k\in\sigma^c} |\langle f, \tilde{f}_k\rangle|^2\right)^{\frac{1}{2}} &= \left(\sum_{k\in\sigma} |\langle f, f_k\rangle|^2 + \sum_{k\in\sigma^c} |\langle f, f_k\rangle - \langle f, f_k - \tilde{f}_k\rangle|^2\right)^{\frac{1}{2}} \\
&\ge \left(\sum_{k\in I} |\langle f, f_k\rangle|^2\right)^{\frac{1}{2}} - \left(\sum_{k\in\sigma^c} |\langle f, f_k - \tilde{f}_k\rangle|^2\right)^{\frac{1}{2}} \\
&\ge \sqrt{A}\|f\| - \|T_{\mathcal{F}J^c} f - T_{\tilde{\mathcal{F}}J^c} f\|
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\sqrt{A} - \|T_{\mathcal{F}J^c} - T_{\tilde{\mathcal{F}}J^c}\|\right) \|f\| \\
&\geq \left(\sqrt{A} - \epsilon\right) \|f\|
\end{aligned}
$$

□

The following theorem is another feature of a result in [1], (Theorem 6.1) in terms of our content.

Theorem 2.2. *Let $\{f_k\}_{k\in I}$ be a frame for H with ϵ-approximation $\{\tilde{f}_k\}_{k\in I}$ such that $\epsilon\sqrt{B}\left(2+\sqrt{\frac{\epsilon}{B}}\right) < \frac{A}{2}$. Then $\{f_k\}_{k\in\sigma} \cup \{\tilde{f}_k\}_{k\in\sigma^c}$ is a frame for H with lower and upper frame bounds $\frac{A}{2}$ and $B(1+(1+\sqrt{\frac{\epsilon}{B}})^2)$ respectively. It means $\{f_k\}_{k\in I}$ and $\{\tilde{f}_k\}_{k\in I}$ are woven.*

3 Woven compact-tight frames

Definition 3.1. [4] We say that a frame is compact-tight if its frame operator S is a compact perturbation of a constant multiple of the identity, that is, $S = cI + K$ with K being a compact operator.

Theorem 3.2. *Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ be a compact-tight frame with bounds A, B and frame operator S. If $\|(1-c)I - K\|^2 \leq \frac{A}{B}$, then $\{f_k\}_{k\in I}$ is woven with $\{Sf_k\}_{k\in I}$.*

Proof. we know that the frame operator S is invertible and so $\{Sf_k\}_{k\in I}$ is a frame. For every $\sigma \subset I$ and $f \in H$

$$
\begin{aligned}
\left(\sum_{k\in\sigma} |\langle f, f_k\rangle)|^2 + |\sum_{k\in\sigma^c} |\langle f, Sf_k\rangle)|^2\right)^{\frac{1}{2}} &= \left(\sum_{k\in\sigma} |\langle f, f_k\rangle)|^2 + |\sum_{k\in\sigma^c} |\langle f, (cI+K)f_k\rangle)|^2\right)^{\frac{1}{2}} \\
&\geq \left(\sum_{k\in\sigma} |\langle f, f_k\rangle)|^2 + \sum_{k\in\sigma^c} |\langle f, f_k\rangle - \langle (I-(cI+K)^* f, f_k\rangle|^2\right) \\
&\geq \left(\sum_{k\in I} |\langle f, f_k\rangle|^2\right)^{\frac{1}{2}} - \left(\sum_{k\in\sigma^c} |\langle ((1-c)I - K^*)f, f_k|^2\rangle\right)^{\frac{1}{2}} \\
&\geq \left(\sqrt{A} - \sqrt{B}\|(1-c)I - K\|\right) \|f\|.
\end{aligned}
$$

Therefore, $\{f_k\}_{k=1}^{\infty} \cup \{Sf_k\}_{k\in I}$ is a frame with lower frame bound

$$
\left(\sqrt{A} - \sqrt{B}\|(1-c)I - K\|\right) > 0
$$

□

It can be deduced from Theorem 3.2 that if $\{f_k\}_{k\in I}$ is a compact-tight frame and $\|(1-c)I - K^{-1}\| \leq \frac{A}{B}$. Then $\{f_k\}_k$ is woven with its canonical dual $\{S^{-1}f_k\}$.

References

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